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A Sharp Asymptotics of the Partition Function for the Collapsed Interacting Partially Directed Self-avoiding Walk

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Abstract

In the present paper, we investigate the collapsed phase of the interacting partially-directed self-avoiding walk (IPDSAW) that was introduced in Zwanzig and Lauritzen (J Chem Phys 48(8):3351, 1968) under a semi-continuous form and later in Binder et al. (J Phys A 23(18):L975–L979, 1990) under the discrete form that we address here. We provide sharp asymptotics of the partition function inside the collapsed phase, proving rigorously a conjecture formulated in Guttmann (J Phys A 48(4):045209, 2015) and Owczarek et al. (Phys Rev Lett 70:951–953, 1993). As a by-product of our result, we obtain that, inside the collapsed phase, a typical IPDSAW trajectory is made of a unique macroscopic bead, consisting of a concatenation of long vertical stretches of alternating signs, outside which only finitely many monomers are lying.

Keywords Polymer collapse · Large deviations · Random walk representation · Local limit theorem

Mathematics Subject Classification Primary 60K35 · Secondary 82B41

Notation

Let $(a_L)_{L\leq 1}$ and $(b_L)_{L\leq 1}$ be two sequences of positive numbers. We write

$$a_L \underset{L\to\infty}{\sim} b_L \quad \text{if} \quad \lim_{L\to\infty} a_L/b_L = 1\,.$$

We also write (const.) to denote generic positive constant whose value may change from line to line.

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1 Introduction

Identifying the behavior of the partition function of a lattice polymer model is in general a challenging question that sparked interest in both the physical and mathematical literature. In a recent survey [10], some polymer models are reviewed that have in common that their partition functions is of the form

$$Q_L \underset{L\to\infty}{\sim} B\mu^L \mu_1^{L^\sigma} L^g\,, \tag{1.1}$$

where B, μ, μ_1, σ, g are real constants depending on the coupling parameters of the model. Among these model, the Interacting Partially Directed Self-Avoiding Walk (referred to under the acronym IPDSAW) which accounts for an homopolymer dipped in a poor (i.e., repulsive) solvent is conjectured to satisfy (1.1) inside its collapsed phase. To be more specific, based on numerics displayed in [10] and in [14] (with simulations up to size $L = 6000$), the values of σ and g are conjectured to be $1/2$ and $-3/4$. In an earlier paper [6,Theorem 2.1], analytic expressions where displayed for μ and μ_1 while σ was proven to be $1/2$. In the present paper, we give a full proof of (1.1) for the IPDSAW in its collapsed phase, in particular we prove that $g = -3/4$ and give an analytic expression of B.

Model

Several mathematical models have been introduced so far in the literature to investigate the collapse transition of a polymer in a repulsive solvant (we refer to [7,Section 6] or [11] for a review on the topic). On the square lattice, a natural pick would be to model the polymer configurations by the trajectories of a self-avoiding walk. However the combinatorial complexity of this walk makes such models very difficult to study. This is the reason why the IPDSAW (that is partially directed) has been introduced, first under a semi-continuous form in [18] and later on under a fully discrete form in [1]. This later version of the model is the one we consider here.

The spatial configurations of the polymer are modeled by the trajectories of a *self-avoiding* random walk on $\mathbb{Z}^2$ that only takes unitary steps *upwards, downwards and to the right* (see Fig. 1). To take into account the monomer-solvent interactions, one considers that, when dipped in a poor solvent, the monomers try to exclude the solvent and therefore attract one another. For this reason, any non-consecutive vertices of the walk though adjacent on the lattice are called *self-touchings* (see Fig. 1) and the interactions between monomers are taken into account by assigning an energetic reward $\beta \geq 0$ to the polymer for each self-touching.

It is convenient to represent the configurations of the model as families of oriented vertical stretches separated by horizontal steps. To be more specific, for a polymer made of $L \in \mathbb{N}$ monomers, the set of allowed path is $\Omega_L := \bigcup_{N=1}^{L} \mathcal{L}_{N,L}$, where $\mathcal{L}_{N,L}$ consists of all families made of N vertical stretches that have a total length $L - N$, that is

$$\mathcal{L}_{N,L} = \Big\{\ell := (\ell_i)_{i=1}^N \in \mathbb{Z}^N : \textstyle\sum_{n=1}^N |\ell_n| + N = L\Big\}. \tag{1.2}$$

With this representation, the modulus of a given stretch corresponds to the number of monomers constituting this stretch (and the sign gives the direction upwards or downwards). For convenience, we require every configuration to end with a horizontal step, and we note that any two consecutive vertical stretches are separated by a step placed horizontally. The latter explains why $\sum_{n=1}^N |\ell_n|$ must equal $L - N$ in order for $\ell = (\ell_i)_{i=1}^N$ to be associated with a polymer made of L monomers (see Fig. 1).

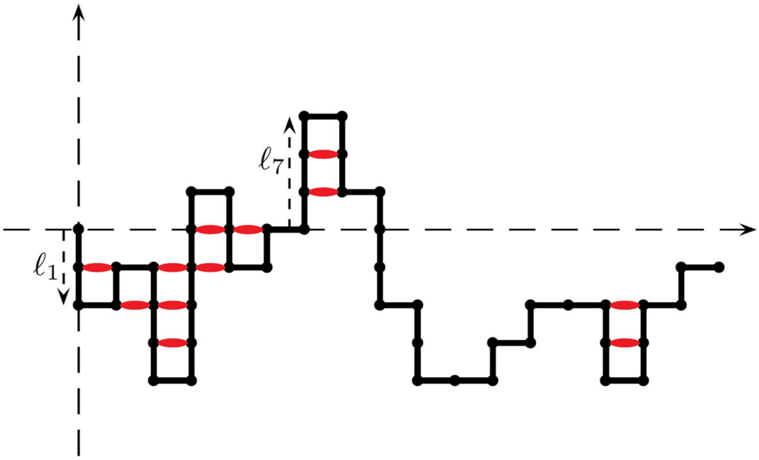

Fig. 1 Representation of an IPDSAW trajectory with length $L = 48$ and horizontal extension $N = 17$. Its self-touchings are shown in red

As mentioned above, the attraction between monomers is taken into account in the Hamiltonian associated with each path $\ell \in \Omega_L$, by rewarding energetically those pairs of consecutive stretches with opposite directions, i.e.,

$$H_{L,\beta}(\ell_1, \ldots, \ell_N) = \beta \sum_{n=1}^{N-1} (\ell_n \,\tilde{\wedge}\, \ell_{n+1}), \tag{1.3}$$

where

$$x \,\tilde{\wedge}\, y = \begin{cases} |x| \wedge |y| & \text{if } xy < 0, \\ 0 & \text{otherwise}, \end{cases} \tag{1.4}$$

and for $x, y \in \mathbb{R}$, $x \wedge y := \min(x, y)$.

One can already note that large values of the Hamiltonian are assigned to trajectories made of few but long vertical stretches with alternating signs. Such paths will be referred to as collapsed configurations. With the Hamiltonian in hand we can define the polymer measure as

$$P_{L,\beta}(\ell) = \frac{e^{H_{L,\beta}(\ell)}}{Z_{L,\beta}}, \quad \ell \in \Omega_L, \tag{1.5}$$

where $Z_{L,\beta}$ is the partition function of the model, i.e.,

$$Z_{L,\beta} = \sum_{N=1}^{L} \sum_{\ell \in \mathcal{L}_{N,L}} e^{H_{L,\beta}(\ell)}. \tag{1.6}$$

2 Main Results

It was first proven in [2] with combinatorial tools that the IPDSAW undergoes a collapse transition at some $\beta_c > 0$, dividing $[0, \infty)$ into an extended phase $\mathcal{E} := [0, \beta_c)$ and a collapsed phase $\mathcal{C} := [\beta_c, \infty)$ (see also [13,Theorem 1.3] for a probabilistic proof). Note that, when the monomer-monomer attraction is switched off ($\beta = 0$), a typical configuration (sampled from $P_{L,\beta}$) has a horizontal extension $O(L)$ since, roughly speaking, every step has a positive probability (bounded from below) to be horizontal. In the extended phase, the interaction intensity β is not yet strong enough to bring this typical horizontal extension from $O(L)$ to $o(L)$. Inside the collapsed phase, in turn, the interaction intensity is large enough to change dramatically the geometric features of a typical trajectory, which roughly looks like a compact ball with a horizontal extension $o(L)$. Asymptotics of $(Z_{L,\beta})_{L\geq 1}$ are displayed and proven in [6,Theorem 2.1 (1) and (2)] for the extended phase and at criticality ($\beta = \beta_c$). Inside the collapsed phase, although the exponential terms of the partition function growth rate were identified via upper and lower bounds (see [6,Theorem 2.1 (iii)]), a full proof of such asymptotics was missing. We close this gap with Theorem 2.1 below, by identifying the polynomial prefactor. This improvement relies on a sharp local limit theorem displayed in Proposition 4.6 for some random walk in a large deviation regime (more precisely it is constrained to be positive, cover a large area and end in 0).

Let us settle some notations that are required to state Theorem 2.1. For $\beta > 0$ we let $\mathbf{P}_\beta$ be the following discrete Laplace probability law on $\mathbb{Z}$:

$$\mathbf{P}_\beta(\cdot = k) = \frac{e^{-\frac{\beta}{2}|k|}}{c_\beta} \quad , \qquad c_\beta := \sum_{k\in\mathbb{Z}} e^{-\frac{\beta}{2}|k|} = \frac{1+e^{-\beta/2}}{1-e^{-\beta/2}}\,. \tag{2.1}$$

Let β_c is the unique positive solution to the equation $\frac{c_\beta}{e^\beta} = 1$ (see [13,Theorem 1.3]); in particular $\frac{c_\beta}{e^\beta} < 1$ for $\beta > \beta_c$. Moreover, we denote by $\mathcal{L}(h)$ the logarithmic moment generating function of Z a random variable of law $\mathbf{P}_\beta$, i.e.,

$$\mathcal{L}(h) \;:=\; \log \mathbf{E}_\beta[e^{hZ}] \;=\; \log\left(\frac{\cosh(\beta/2)-1}{\cosh(\beta/2)-\cosh(h)}\right), \qquad h \in \big(-\tfrac{\beta}{2}, \tfrac{\beta}{2}\big), \tag{2.2}$$

which is smooth, strictly convex and even on $(-\beta/2, \beta/2)$, and $\mathcal{L}''$ is bounded away from 0 on $(-\beta/2, \beta/2)$ by some positive constant.

We now display some definitions required to study the asymptotics of the probability that some random walk trajectory encloses an atypically large area (which were first introduced in [8]), as well as the escape probability of a certain class of drifted random walks.

Asymptotics for trajectories enclosing an atypically large area. We let $\mathcal{G} : (-\beta, \beta) \to \mathbb{R}$ be defined as

$$\mathcal{G}(h) := \int_0^1 \mathcal{L}(h(\tfrac{1}{2}-s))\mathrm{d}s, \quad \text{for} \quad h \in (-\beta, \beta). \tag{2.3}$$

Recalling (2.2), a straightforward computation yields that $\mathcal{G}''(h) > 0$ for $h \in (-\beta, \beta)$, and $\mathcal{G}'$ is a $\mathcal{C}^1$ diffeomorphism from $(-\beta, \beta)$ to $\mathbb{R}$. We let $q \in \mathbb{R} \mapsto \widetilde{h}^q$ be its inverse function. Considering $X := (X_i)_{i\in\mathbb{N}}$ a random walk starting from the origin, whose increments $(X_i - X_{i-1})_{i\in\mathbb{N}}$ are i.i.d. with law $\mathbf{P}_\beta$, we denote by $A_N(X)$ (or A_N when there is no risk of confusion) the algebraic area enclosed by X up to time N, i.e.,

$$A_N(X) := X_1 + \cdots + X_N. \tag{2.4}$$

It is proven in [8] that, for $q \in (0,\infty) \cap \frac{\mathbb{N}}{N^2}$, the exponential decay rate of the event $\{A_N = qN^2, X_N = 0\}$ is given by the Legendre transform of $\mathcal{G}$ (or "convex dual", see [16,Sec. 12]), that is $\widetilde{\psi}$ defined by

$$\widetilde{\psi}(q) \;:=\; \sup\{qh - \mathcal{G}(h),\, h \in (-\beta,\beta)\} \;=\; q\,\widetilde{h}^q - \mathcal{G}(\widetilde{h}^q), \quad \text{for} \quad q \in (0,\infty), \tag{2.5}$$

(notice that $\mathcal{G}$ is strictly convex, hence so is $\widetilde{\psi}$), and in this large deviation regime, the fluctuations of $(\frac{1}{N}A_N(X), X_N)$ around $(qN, 0)$ are asymptotically Gaussian (see also [8,Section 4] and Appendix A.1 for more details). The determinant of the covariance matrix of this limiting distribution is given by

$$\vartheta(h) = \int_0^1 s^2 \mathcal{L}''[h(s-\tfrac{1}{2})]\,ds \int_0^1 \mathcal{L}''[h(s-\tfrac{1}{2})]ds - \left[\int_0^1 s\,\mathcal{L}''[h(s-\tfrac{1}{2})]ds\right]^2, \quad (2.6)$$

which is positive by the Cauchy–Schwarz inequality.

Tilting and escape probability. For $|h| < \beta/2$ we let $\widetilde{\mathbf{P}}_h$ be the probability law on $\mathbb{Z}$ defined by perturbing $\mathbf{P}_\beta$ as

$$\frac{\mathrm{d}\widetilde{\mathbf{P}}_h}{\mathrm{d}\mathbf{P}_\beta}(k) = e^{hk-\mathcal{L}(h)} \quad k \in \mathbb{Z}. \quad (2.7)$$

For $h \in (0, \beta/2)$, we consider a random walk $X := (X_i)_{i\in\mathbb{N}}$ such that $X_0 = 0$ and whose increments $(X_i - X_{i-1})_{i\in\mathbb{N}}$ are i.i.d. with law $\widetilde{\mathbf{P}}_h$, i.e. with a positive drift (with an abuse of notation, $\widetilde{\mathbf{P}}_h$ will also denote the law of this random walk). We denote by $\kappa(h)$ the probability that X never returns to the lower half plane, that is

$$\kappa(h) := \widetilde{\mathbf{P}}_h\big(X_i > 0, \ \ \forall i \in \mathbb{N}\big) = \frac{e^{2h}-1}{e^{h+\beta/2}-1}, \quad (2.8)$$

where the second identity is proven in Lemma 5.15.

Finally, we compute in Proposition 4.6 below an equivalent as $N \to \infty$ to the probability of the event $\{A_N = qN^2, X_N = 0, X_i > 0\ \forall\, 1 \le i \le N-1\}$, which involves the constant prefactor

$$C_{\beta,q} := \frac{1}{2\pi\ \vartheta(\widetilde{h}^q)^{\frac{1}{2}}}\ \kappa\big(\tfrac{\widetilde{h}^q}{2}\big)^2, \quad (2.9)$$

for $\beta, q > 0$. With all those definitions, we may now state our main result.

Theorem 2.1 *For any $\beta > \beta_c$, one has*

$$Z_{L,\beta} \underset{L\to\infty}{\sim} \frac{K_\beta}{L^{3/4}}\, e^{\beta L + \widetilde{\mathcal{G}}(a_\beta)\sqrt{L}}, \quad (2.10)$$

with

$$\widetilde{\mathcal{G}}(x) \ := \ x \log \frac{c_\beta}{e^\beta} - x\,\widetilde{\psi}(x^{-2}), \quad x > 0, \quad (2.11)$$

and

$$a_\beta \ := \ \arg\max\big\{\widetilde{\mathcal{G}}(x)\colon x \in]0,\infty[\big\}, \quad (2.12)$$

and

$$K_\beta := \frac{2\sqrt{2\pi}\, C_{\beta,a_\beta^{-2}}\, e^{\widetilde{\psi}'(a_\beta^{-2})}}{\big[(1+e^{-\beta})e^{\operatorname{arccosh}(e^{-\beta/2}\cosh(\beta))} - e^{\beta/2}(1-e^{-\beta})\big]^2\, a_\beta^2\, \big|\widetilde{\mathcal{G}}''(a_\beta)\big|^{1/2}}. \quad (2.13)$$

Theorem 2.1 is proven in Sect. 4. The proof requires to decompose a trajectory into a succession of *beads* that are sub-trajectories made of non-zero vertical stretches of alternating signs (see Sect. 3 for a rigorous definition). Inside the collapsed phase, an issue raised by physicists was to understand whether a typical trajectory contains a unique macroscopic bead or not. Thus, for every $\ell \in \Omega_L$ we let N_ℓ be its horizontal extension (i.e., $\ell \in \mathcal{L}_{N_\ell,L}$) and also $|I_{\max}(\ell)|$ be the length of its largest bead, i.e.,

$$|I_{\max}(\ell)| := \max\big\{ \textstyle\sum_{i=u}^{v}(1+|\ell_i|)\colon\ 1 \le u \le v \le N_\ell,\ \ell_i\ell_{i+1} < 0\ \ \forall u \le i \le v-1\big\}. \quad (2.14)$$

With [4,Theorem C] it is known that a typical trajectory indeed contains a unique macroscopic bead, and that at most $(\log L)^4$ monomers lay outside this large bead. We improve this result with the following theorem, by showing that only finitely many monomers are to be found outside the unique macroscopic bead. Recall from (1.5) that $P_{L,\beta}$ denotes the polymer measure.

Theorem 2.2 *For any* $\beta > \beta_c$,

$$\lim_{k\to\infty} \liminf_{L\to\infty} P_{L,\beta}\big(|I_{\max}(\ell)| \geq L - k\big) = 1. \tag{2.15}$$

Comments on the Results

Let us give some insight into (2.10–2.13). In Proposition 4.1, we provide sharp asymptotics to the partition function of the *single-bead* version of the model, in which vertical stretches are constrained to alternate signs and be non-zero. Since any trajectory of the full model may be decomposed into a sequence of single-bead ones (see Sect. 3.1 below), we prove in Sect. 4 with a renewal argument that the same asymptotics hold for the full model, up to an explicit constant factor.

Regarding the single-bead model, we display in Sect. 3.2 a probabilistic representation of its partition function, which was initially introduced in [13]. Let X be a random walk with increments distributed as $\mathbf{P}_\beta$ (recall (2.1)). For a length $L \in \mathbb{N}$ and when fixing a horizontal extension $N \in \mathbb{N}$, the single-bead partition function matches the probability that X satisfies the event $\{A_N = L - N, X_{N+1} = 0, X_i > 0 \;\forall\, 1 \leq i \leq N\}$, up to an explicit factor $2c_\beta e^{\beta L}(c_\beta e^{-\beta})^N$ (see (3.16)). This probability matches the partition function of a 1-dimensional SOS-model (Solid-On-Solid) with additional constraints: on the one hand the trajectory must enclose a fixed area and end in 0, as in the SOS-models previously studied in [8, 12]; on the other hand the walk has to remain positive. Therefore, we prove Proposition 4.1 by:

- First observing that, in the Collapsed regime, the main contribution to the single-bead partition function comes from trajectories with horizontal extension $N = x\sqrt{L}$, $x \in [x_1, x_2] \cap \frac{\mathbb{N}}{\sqrt{L}}$ (see [4,Lemma 4.4]). For those trajectories, the latter event becomes $\{A_N = qN^2 + O(N), X_{N+1} = 0, X_i > 0 \;\forall\, 1 \leq i \leq N\}$ with $q = x^{-2} > 0$, which is a large deviation regime for the walk X.
- Then providing an exact equivalent for the probability of the latter event in Proposition 4.6, which is the main feature of this paper. The case without the positivity constraint has already been handled in [8] (see also [12] where the authors compute the exponential decay rate in N, which is $\widetilde{\psi}(q)$), and our result strongly improves previous estimates such as [4,Prop. 2.4–2.5], by deriving sharply the effect of the positivity constraint on this equivalent (whereas [4] only computes polynomial bounds).

Finally, Theorem 2.1 is obtained by optimizing on admissible ratios $\frac{N}{\sqrt{L}} = x \in [x_1, x_2]$ in the partition function. In particular, notice that $\widetilde{\mathcal{G}}(a_\beta)$ is the Legendre transform of the (strictly convex) rate function $x \mapsto x\,\widetilde{\psi}(x^{-2})$, evaluated at $\log(c_\beta e^{-\beta})$.

Outline of the Paper

With Sect. 3 below, we introduce some mathematical tools of particular importance for the rest of the paper. Thus, in Sect. 3.1 we define rigorously the set containing the single-bead trajectories. Such beads allow us to decompose any paths in Ω_L into sub-trajectories that are not interacting with each other. Under the polymer measure, the accumulated lengths of those beads form a renewal process which is of key importance throughout the paper. Section 3.2 is dedicated to the random walk representation of the model and provides a probabilistic expression for many partition functions introduced in the paper. In Sect. 4 we prove Theorem 2.1 subject to Proposition 4.1 which gives sharp asymptotics for those partition functions associated with single bead trajectories. Proposition 4.1 is proven afterwards subject to Proposition 4.6. The proof of Proposition 4.6 is divided into 4 steps, displayed in Sect. 5 after we introduce a tilted law for the random walk X inspired from [8], for which the event $\{A_N(X) = q N^2, X_N = 0\}$ becomes typical. Note that Steps 3 and 4 from Sect. 5 require the local limit theorem which is displayed in Appendix A.1, and Proposition 5.4 is proven in Appendix A.2. Finally, Theorem 2.2 is proven in Sect. 6 by using mostly the asymptotics provided by Theorem 2.1.

3 Preparations

3.1 One Bead Trajectories and Renewal Structure

We call *bead* a maximal succession of non-zero vertical stretches with alternate signs; more precisely a bead terminates whenever the trajectory makes a zero-length vertical stretch, or two non-zero stretches of the same sign. Any trajectory $\ell \in \Omega_L$ can be decomposed into a succession of beads, and this decomposition is unique.

Let $L \in \mathbb{N}$ and denote by Ω_L° the subset of Ω_L gathering single-bead trajectories of length L, i.e. trajectories made of non-zero vertical stretches that alternate orientations. Thus, we set $\Omega_L^\circ = \cup_{N=1}^{L/2} \mathcal{L}_{N,L}^\circ$ with

$$\mathcal{L}_{N,L}^\circ := \Big\{ (\ell_i)_{i=1}^N \in \mathbb{Z}^N : \sum_{i=1}^N |\ell_i| = L - N, \quad \ell_i \ell_{i+1} < 0 \quad \forall 1 \le i < N \Big\}. \tag{3.1}$$

We also denote by $Z_{L,\beta}^\circ$ the partition function restricted to trajectories in Ω_L°, i.e.,

$$Z_{L,\beta}^\circ := \sum_{N=1}^{L/2} \sum_{\ell \in \mathcal{L}_{N,L}^\circ} e^{\beta H(\ell)}. \tag{3.2}$$

The bead decomposition and single-bead model were already introduced in [4]; however additional care is required to derive sharp asymptotics of $Z_{L,\beta}$: after a zero-length vertical stretch, the sign of the first stretch of the next bead is unrestricted, whereas it is otherwise constrained by the sign of the last stretch of the previous bead. To bypass this combinatorial inconvenience, we tweak the decomposition to integrate the zero-length stretches at the beginning of beads (see Fig. 2), giving rise to a renewal structure: the realization of a single-bead trajectory has no influence over the value of the partition function associated with the next bead. We define $\widehat{\Omega}^\circ$ the set of *extended beads* as

$$\widehat{\Omega}^\circ := \bigcup_{L \ge 2} \widehat{\Omega}_L^\circ, \tag{3.3}$$

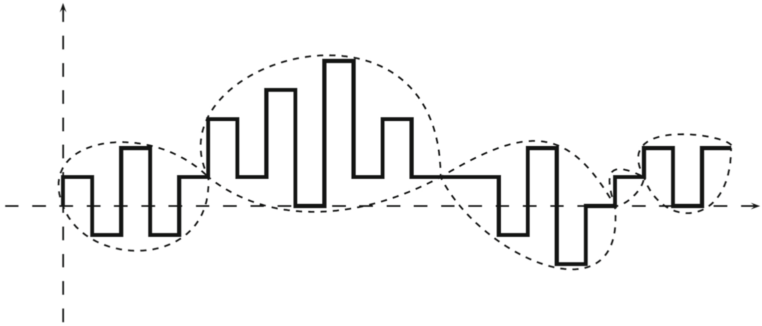

Fig. 2 Decomposition of an IPDSAW trajectory into "extended" beads, see (3.3-3.5). A new bead begins whenever two non-zero length stretches have the same sign, or when there is a succession of zero-length vertical stretches

where $\widehat{\Omega}_L^\circ$ is the subset of Ω_L gathering trajectories which may or may not start with a sequence of zero-length stretches and form subsequently a unique bead, i.e.,

$$\widehat{\Omega}_L^\circ := \bigcup_{k=0}^{L-2} \widehat{\Omega}_L^{\circ,k}, \tag{3.4}$$

with

$$\begin{aligned} \widehat{\Omega}_L^{\circ,0} &:= \Big\{\ell \in \Omega_L^\circ \colon \ell_1 > 0\Big\}, \\ \widehat{\Omega}_L^{\circ,k} &:= \Big\{\ell \in \Omega_L \colon N_\ell > k, \ell_1 = \cdots = \ell_k = 0, (\ell_{i+k})_{i=1}^{N_\ell - k} \in \Omega_{L-k}^\circ\Big\}, \quad k \in \{1, \ldots, L-2\}. \end{aligned} \tag{3.5}$$

We recall (3.2) and (3.4) and the partition function restricted to trajectories in $\widehat{\Omega}_L^\circ$ becomes:

$$\widehat{Z}_{L,\beta}^\circ := \sum_{k=0}^{L-2} \Big[\frac{1}{2}\, 1_{\{k=0\}}\, Z_{L,\beta}^\circ + 1_{\{k \in \mathbb{N}\}}\, Z_{L-k,\beta}^\circ\Big]. \tag{3.6}$$

Note that, in the definition of $\widehat{\Omega}_L^{\circ,0}$ in (3.5) the sign of ℓ_1 is prescribed because if an extended bead does not start with a zero-length stretch, then the sign of its first stretch must be the same as that of the last stretch of the preceding bead; hence the factor $\frac{1}{2}$ in (3.6) (see Fig. 2, where the 2nd, 4th and 5th beads start with non-zero stretches with constrained signs, and the 3rd bead starts with two zero-length stretches). Of course this latter restriction does not apply to the very first extended bead of a trajectory and this is why we define

$$\bar{Z}_{L,\beta}^\circ := \sum_{k=0}^{L-2} Z_{L-k,\beta}^\circ. \tag{3.7}$$

For convenience, we also define Ω_L^c the subset of Ω_L containing trajectories ending with a non-zero stretch, i.e.,

$$\Omega_L^c := \{\ell \in \Omega_L \colon\ \ell_{N_\ell} \neq 0\}, \tag{3.8}$$

(see Fig. 2 for an example). At this stage we can decompose a given trajectory $\ell \in \Omega_L^c$ into extended beads by cutting the trajectory at times $(\tau_j)_{j=0}^{n(\ell)}$ defined as $\tau_0 = 0$ and for $j \in \mathbb{N}$ such that $\tau_{j-1} < N_\ell$

$$\tau_j := \max\Big\{s > \tau_{j-1} \colon\ (\ell_i)_{i=1+\tau_{j-1}}^s \in \widehat{\Omega}^\circ \quad \text{or} \quad (-\ell_i)_{i=1+\tau_{j-1}}^s \in \widehat{\Omega}^\circ\Big\}. \tag{3.9}$$

Then $n(\ell)$ is the number of beads composing ℓ and satisfies $\tau_{n(\ell)} = N_\ell$, thus

$$\ell = \odot_{j=1}^{n(\ell)} \mathcal{B}_j \quad \text{with} \quad \mathcal{B}_j := (\ell_{\tau_{j-1}+1}, \dots, \ell_{\tau_j}), \tag{3.10}$$

where $\odot$ denotes the concatenation. We also set $\mathfrak{X}_0 = 0$ and for $j \in \{1, \dots, n_\ell\}$, we denote by $\mathfrak{X}_j - \mathfrak{X}_{j-1}$ the number of steps (or monomers) that the j-th bead is made of (also referred to as *total length* of the bead), that is,

$$\mathfrak{X}_j - \mathfrak{X}_{j-1} = \tau_j - \tau_{j-1} + |\ell_{\tau_{j-1}+1}| + \cdots + |\ell_{\tau_j}|, \quad j \in \{1, \dots, n_\ell\}. \tag{3.11}$$

The set $\mathfrak{X} := \{0, \mathfrak{X}_1, \dots, \mathfrak{X}_{n_\ell}\}$ contains the accumulated lengths of the beads forming ℓ, in particular $\mathfrak{X}_{n_\ell} = L$.

Remark 3.1 Note that a trajectory $\ell \in \Omega_L \setminus \Omega_L^c$ may also be decomposed into extended beads as in (3.9–3.11). The only difference is that the very last bead is followed by a sequence of zero-length vertical stretches, i.e., $\mathfrak{X}_{n_\ell} = L - k$ for some $k \in \{1, \dots, L\}$ and the last k vertical stretches in ℓ have zero-length.

By using this bead-decomposition we can rewrite $Z_{L,\beta}^c$ the partition function restricted to Ω_L^c as

$$\begin{aligned} Z_{L,\beta}^c &= \sum_{r=1}^{L/2} \sum_{t_1+\cdots+t_r=L} Z_{L,\beta}^c\big(n(\ell) = r, \mathfrak{X}_i - \mathfrak{X}_{i-1} = t_i, \forall 1 \le i \le r\big) \\ &= \sum_{r=1}^{L/2} \sum_{t_1+\cdots+t_r=L} \bar{Z}_{t_1,\beta}^{\circ} \prod_{j=2}^{r} \widehat{Z}_{t_j,\beta}^{\circ}, \end{aligned} \tag{3.12}$$

where $Z_{L,\beta}^c(D)$ denotes the partition function restricted to trajectories $\ell \in \Omega_L^c$ having property D.

3.2 Probabilistic Representation

The aim of this section is to give a probabilistic expression of the partition function $Z_{L,\beta}^{\circ}$ and to use it subsequently to provide closed expression of the generating functions associated with $(\widehat{Z}_{L,\beta}^{\circ})_{L\ge 2}$ and with $(\bar{Z}_{L,\beta}^{\circ})_{L\ge 2}$.

We recall the definition of $\mathbf{P}_\beta$ and c_β from (2.1), and for $x \in \mathbb{Z}$ we denote by $\mathbf{P}_{\beta,x}$ the law of a random walk $X := (X_i)_{i\ge 0}$ starting from x (i.e., $X_0 = x$) and such that $(X_{i+1} - X_i)_{i\ge 0}$ is an i.i.d. sequence of random variables with law $\mathbf{P}_\beta$. In the case $x = 0$, we omit the x-dependence of $\mathbf{P}_{\beta,x}$ when there is no risk of confusion. We also recall from (2.4) that A_N defines the algebraic area enclosed in-between a random walk trajectory and the x-axis after N steps.

Recall (3.1) and (3.2), and let us now briefly remind the transformation that allows us to give a probabilistic representation of $Z_{L,\beta}^{\circ}$ (we refer to [5] for a review on the recent progress made on IPDSAW by using probabilistic tools). First, note that for $x, y \in \mathbb{Z}$ one can write $x \,\tilde{\wedge}\, y = \frac{1}{2}(|x| + |y| - |x + y|)$. By using the latter equality to compute the Hamiltonian (recall (1.3)), we may rewrite (3.2) as

$$Z^{\circ}_{L,\beta} = \sum_{N=1}^{L/2} \sum_{\substack{\ell \in \mathcal{L}^{\circ}_{N,L} \\ \ell_0=\ell_{N+1}=0}} \exp\Big(\beta \sum_{n=1}^{N} |\ell_n| - \tfrac{\beta}{2} \sum_{n=0}^{N} |\ell_n + \ell_{n+1}|\Big)$$
$$= c_\beta\, e^{\beta L} \sum_{N=1}^{L/2} \Big(\tfrac{c_\beta}{e^\beta}\Big)^N \sum_{\substack{\ell \in \mathcal{L}^{\circ}_{N,L} \\ \ell_0=\ell_{N+1}=0}} \prod_{n=0}^{N} \frac{\exp\Big(-\frac{\beta}{2}|\ell_n + \ell_{n+1}|\Big)}{c_\beta}. \tag{3.13}$$

Henceforth, for convenience, we assume that any $\ell \in \mathcal{L}^{\circ}_{N,L}$ satisfies $\ell_0 = \ell_{N+1} = 0$. At this stage, we denote by B_N^+ the set of those N-step integer-valued random walk trajectories, starting and ending at 0 and remaining positive in-between, i.e.,

$$B_N^+ := \big\{(x_i)_{i=0}^N \in \mathbb{Z}^{N+1} :\ x_0 = x_N = 0,\ x_i > 0\ \forall 0 < i < N\big\}. \tag{3.14}$$

It remains to notice that the map

$$T_N : \begin{array}{ccc} \{\ell \in \mathcal{L}^{\circ}_{N,L} :\ \ell_1 > 0\} & \to & \big\{(x_i)_{i=0}^{N+1} \in B_{N+1}^+ :\ A_N(x) = L - N\big\} \\ (\ell_i)_{i=0}^{N+1} & \mapsto & (x_i)_{i=0}^{N+1} := ((-1)^{i-1}\, \ell_i)_{i=0}^{N+1} \end{array}, \tag{3.15}$$

is a one-to-one correspondence, and that for $\ell \in \mathcal{L}^{\circ}_{N,L}$ the increments of $T_N(\ell)$ are in modulus equal to $(|\ell_{i-1} + \ell_i|)_{i=1}^{N+1}$. Therefore, set $\Gamma_\beta := c_\beta / e^\beta$ and (3.13) becomes

$$Z^{\circ}_{L,\beta}\, e^{-\beta L} = 2\, c_\beta \sum_{N=1}^{L/2} (\Gamma_\beta)^N\, \mathbf{P}_\beta\big(X \in B_{N+1}^+,\ A_N(X) = L - N\big), \tag{3.16}$$

where the factor 2 in the r.h.s. in (3.16) is required to take into account those $\ell \in \mathcal{L}^{\circ}_{N,L}$ satisfying $\ell_1 < 0$.

With the next Lemma, we provide an explicit expression of the generating functions $\sum_{L\geq 2} \widehat{Z}^{\circ}_{L,\beta}\, z^L$ and $\sum_{L\geq 2} \bar{Z}^{\circ}_{L,\beta}\, z^L$ at $z = e^{-\beta}$. This is needed to determine K_β in Theorem 2.1.

Lemma 3.2 *For $\beta > 0$,*

$$\delta_1(\beta) := \sum_{L\geq 2} \bar{Z}^{\circ}_{L,\beta}\ e^{-\beta L} = \frac{2\, e^\beta}{1 - e^{-\beta}} r_\beta\,,$$
$$\delta_2(\beta) := \sum_{L\geq 2} \widehat{Z}^{\circ}_{L,\beta}\ e^{-\beta L} = e^\beta c_{2\beta}\, r_\beta\,, \tag{3.17}$$

with $r_\beta := \mathbf{E}_\beta\big[1_{\{X_1>0\}}\, 1_{\{X_\rho=0\}}\, (\Gamma_\beta)^\rho\big]$ where $\rho := \inf\{i \geq 1 :\ X_i \leq 0\}$. Moreover,

$$r_\beta = \begin{cases} +\infty & \text{if } \beta < \beta_c\,, \\ 1 - e^{-\beta} - e^{-\frac{\beta}{2} + \operatorname{arccosh}\left(e^{-\beta/2}\cosh(\beta)\right)} & \text{if } \beta \geq \beta_c\,. \end{cases} \tag{3.18}$$

Proof We recall (3.16) and we start by computing

$$\sum_{L\geq 2} Z^{\circ}_{L,\beta}\, e^{-\beta L} = 2\, c_\beta \sum_{N\geq 1} (\Gamma_\beta)^N \sum_{L\geq 2N} \mathbf{P}_\beta\big(X \in B_{N+1}^+,\ A_N(X) = L - N\big)$$
$$= 2\, c_\beta \sum_{N\geq 1} (\Gamma_\beta)^N\, \mathbf{P}_\beta\big(X \in B_{N+1}^+\big)$$
$$= 2\, e^\beta\, r_\beta. \tag{3.19}$$

Then, we use (3.6) to write

$$\sum_{L\geq 2} \widehat{Z}^{\circ}_{L,\beta}\, e^{-\beta L} := \sum_{L\geq 2}\sum_{k=0}^{L-2} e^{-\beta L}\Big[\frac{1}{2}\, 1_{\{k=0\}}\, Z^{\circ}_{L,\beta} + 1_{\{k\in\mathbb{N}\}}\, Z^{\circ}_{L-k,\beta}\Big]$$
$$= \frac{1}{2}\sum_{L\geq 2} e^{-\beta L} Z^{\circ}_{L,\beta} + \sum_{k=1}^{\infty} e^{-\beta k}\sum_{t=2}^{\infty} e^{-\beta t} Z^{\circ}_{t,\beta}$$
$$= \Big(\frac{1}{2} + \frac{e^{-\beta}}{1-e^{-\beta}}\Big)\sum_{L\geq 2} e^{-\beta L} Z^{\circ}_{L,\beta}, \tag{3.20}$$

and it remains to combine (3.19) and (3.20) to obtain the second equality in (3.17). Using (3.7), the very same computation allows us to obtain the first equality in (3.17).

To obtain a formula for r_β, let us now compute $\mathbf{E}_\beta[(\Gamma_\beta)^\rho]$: indeed, the fact that the increments of X follow a discrete Laplace law entails that $(\rho, X_1, \dots, X_{\rho-1})$ and X_ρ are independent, and moreover that $-X_\rho$ follows a geometric law on $\mathbb{N}\cup\{0\}$ with parameter $1-e^{-\beta/2}$. Thereby,

$$\mathbf{E}_\beta\big[(\Gamma_\beta)^\rho\big] = \mathbf{E}_\beta\big[1_{\{X_1>0\}}(\Gamma_\beta)^\rho\big] + \Gamma_\beta\, \mathbf{P}_\beta(X_1 \leq 0)$$
$$= \frac{r_\beta}{\mathbf{P}_\beta(-X_\rho = 0)} + \frac{c_\beta}{e^\beta}\left(\frac{1}{c_\beta} + \frac{1}{2}\Big(1 - \frac{1}{c_\beta}\Big)\right)$$
$$= \frac{r_\beta + e^{-\beta}}{1-e^{-\beta/2}}. \tag{3.21}$$

It is a straightforward application of [3,Eq. (4.5)] that there exists $c > 0$ depending on β only such that

$$\mathbf{P}_\beta(\rho = t) \sim \frac{c}{t^{3/2}} \qquad \text{as} \quad t\to\infty\,. \tag{3.22}$$

Recall that $\beta \mapsto \Gamma_\beta$ is decreasing on $(0,\infty)$; in particular when $\beta < \beta_c$, one has $\Gamma_\beta > 1$, so (3.21) and (3.22) imply $r_\beta = +\infty$.

Recall the definition and expression of $\mathcal{L}$ from (2.2). When $\beta \geq \beta_c$, notice that $(e^{-\zeta X_n + \log(\Gamma_\beta)n})_{n\geq 0}$ is a martingale under $\mathbf{P}_{\beta,0}$ if and only if $\mathcal{L}(-\zeta) = -\log(\Gamma_\beta)$. Since $\Gamma_\beta \leq 1$ and $\mathcal{L}$ is decreasing, not bounded on $(-\beta/2, 0]$, there is a unique $\zeta_\beta \in [0,\beta/2)$ satisfying this equality, given by

$$\zeta_\beta = \operatorname{arccosh}\big((1-\Gamma_\beta)\cosh(\beta/2) + \Gamma_\beta\big) = \operatorname{arccosh}\big(e^{-\beta/2}\cosh(\beta)\big). \tag{3.23}$$

Noticing that this martingale stopped at time ρ is uniformly integrable and using Doob's Optional Stopping Theorem, we finally obtain

$$\mathbf{E}_\beta\big[(\Gamma_\beta)^\rho\big] = \mathbf{E}_\beta\Big[e^{-\zeta_\beta X_\rho}\Big]^{-1} = \frac{1-e^{\zeta_\beta-\beta/2}}{1-e^{-\beta/2}}, \tag{3.24}$$

(recall that $-X_\rho$ follows a geometric law on $\mathbb{N}\cup\{0\}$ with parameter $1-e^{-\beta/2}$), which concludes the proof by recollecting (3.21) and (3.23). □

To end this section we state and prove the following corollary which is needed in the proof of Theorem 2.1 (see Sect. 4) below.

Corollary 3.3 *For any $\beta > \beta_c$, we have $\delta_2(\beta) < 1$.*

Proof We observe that $\beta \to \delta_2(\beta)$ is decreasing simply because for every $L \geq 2$ and every $\ell \in \Omega_L$ the quantity $\beta \to H_{L,\beta}(\ell) - \beta L$ is decreasing (recall (1.3), and notice that any trajectory $\ell \in \Omega_L$ realizes less than L self-touchings). Therefore, it only remains to prove that $\delta_2(\beta_c) = 1$, and the corollary follows. Recall that $\Gamma_{\beta_c} = 1$, which ensures that $e^{\beta_c/2}$ is a solution to $X^3 - X^2 - X - 1 = 0$ and that $\zeta_{\beta_c} = 0$ in (3.23). This implies both

$$r_{\beta_c} = 1 - e^{-\beta_c} - e^{-\beta_c/2} = e^{-3\beta_c/2},$$

and

$$\delta_2(\beta_c) = e^{-\beta_c/2}\left(\frac{1+e^{-\beta_c}}{1-e^{-\beta_c}}\right) = 1,$$

which concludes the proof. □

4 Proof of Theorem 2.1

In this section we provide a sharp estimate for the partition function of single-bead trajectories in Proposition 4.1, with which we prove Theorem 2.1. The proof of Proposition 4.1 is displayed afterwards subject to Proposition 4.6 and Lemma 4.5. Recall the definition of $\widetilde{\mathcal{G}}$ from (2.11), and let us point out that it was proven in [4,(1.27)] that $\widetilde{\mathcal{G}}$ is smooth, concave, negative on $(0, \infty)$, and it admits a unique maximizer $a_\beta \in (0, \infty)$.

Proposition 4.1 *For $\beta > \beta_c$, there exists $K_\beta^\circ > 0$ such that*

$$Z_{L,\beta}^\circ \underset{L\to\infty}{\sim} \frac{K_\beta^\circ}{L^{3/4}} e^{\beta L + \widetilde{\mathcal{G}}(a_\beta)\sqrt{L}}. \tag{4.1}$$

Corollary 4.2 *For $\beta > \beta_c$, there exist $\widehat{K}_\beta > 0$ and $\bar{K}_\beta > 0$ such that*

$$\widehat{Z}_{L,\beta}^\circ \underset{L\to\infty}{\sim} \frac{\widehat{K}_\beta}{L^{3/4}} e^{\beta L + \widetilde{\mathcal{G}}(a_\beta)\sqrt{L}} \quad \textit{and} \quad \bar{Z}_{L,\beta}^\circ \underset{L\to\infty}{\sim} \frac{\bar{K}_\beta}{L^{3/4}} e^{\beta L + \widetilde{\mathcal{G}}(a_\beta)\sqrt{L}}. \tag{4.2}$$

Explicit formulae for $\widehat{K}_\beta$, $\bar{K}_\beta$ and K_β° are given below in (4.4) and (4.37). Let us point out that (4.1) is a substantial improvement of [4,Prop. 4.2], where the polynomial factors were $\frac{1}{L^\kappa}$ with $\kappa > 1$ in the lower bound, and $\frac{1}{\sqrt{L}}$ in the upper bound. Moreover Corollary 4.2 is a straightforward consequence of Proposition 4.1. Indeed, let us define for convenience

$$h(n) := \frac{1}{n^{3/4}} e^{\widetilde{\mathcal{G}}(a_\beta)\sqrt{n}}, \quad n \in \mathbb{N}. \tag{4.3}$$

Recollecting (3.6), we write

$$\frac{e^{-\beta L}}{h(L)} \widehat{Z}_{L,\beta}^\circ = \frac{1}{2}\frac{e^{-\beta L}}{h(L)} Z_{L,\beta}^\circ + \sum_{k=1}^{L-2} e^{-\beta k} \frac{e^{-\beta(L-k)}}{h(L)} Z_{L-k,\beta}^\circ,$$

and similarly for (3.7). Notice that $h(L) \sim h(L-k)$ as $L \to \infty$ for any $k \in \mathbb{N}$, so (4.1) and dominated convergence imply that (4.2) holds true with

$$\widehat{K}_\beta = K_\beta^\circ \frac{1+e^{-\beta}}{2(1-e^{-\beta})} \quad \text{and} \quad \bar{K}_\beta = K_\beta^\circ \frac{1}{1-e^{-\beta}}. \tag{4.4}$$

Proof of Theorem 2.1 subject to Proposition 4.1 Recall the definitions of $\delta_1(\beta)$ and $\delta_2(\beta)$ in (3.17), and define two probability laws q_1 and q_2 on $\mathbb{N}$ by

$$q_1(t) := \delta_1(\beta)^{-1}\, \bar{Z}^{\circ}_{t,\beta}\, e^{-\beta t} \quad t \geq 2\,, \tag{4.5}$$

$$q_2(t) := \delta_2(\beta)^{-1}\, \widehat{Z}^{\circ}_{t,\beta}\, e^{-\beta t} \quad t \geq 2\,. \tag{4.6}$$

For $r \in \mathbb{N}$, we denote by $q_2^{r\,*}$ the convolution product of r times q_2 and by $q_1 * q_2^{r*}$ the convolution product between q_1 and $q_2^{r\,*}$. This allows us to rewrite (3.12) under the form

$$\widetilde{Z}^{c}_{L,\beta} := e^{-\beta L} Z^{c}_{L,\beta} = \frac{\delta_1(\beta)}{\delta_2(\beta)} \sum_{r\geq 1} \delta_2(\beta)^r \left[q_1 * q_2^{(r-1)*}\right](L)\,. \tag{4.7}$$

Recalling (4.3), Corollary 4.2 implies that

$$\begin{aligned} q_1(n) &\underset{n\to\infty}{\sim} u_\beta\, h(n), \quad \text{with} \quad u_\beta := \frac{\bar{K}_\beta}{\delta_1(\beta)}\,, \\ q_2(n) &\underset{n\to\infty}{\sim} v_\beta\, h(n), \quad \text{with} \quad v_\beta := \frac{\widehat{K}_\beta}{\delta_2(\beta)}\,. \end{aligned} \tag{4.8}$$

At this stage, Theorem 2.1 is a straightforward consequence of Claims 4.3 and 4.4 below. Those claims are proven in [9,Corollary 4.13 and Theorem 4.14] in the case where $q_1 \equiv q_2$. However, (4.8) guarantees that the proof in [9] can easily be adapted to our case since q_1 and q_2 enjoy the same asymptotic behavior (up to a constant).

Claim 4.3 *For $\beta > 0$ and $r \in \mathbb{N} \cup \{0\}$ it holds that $q_1 * q_2^{r*}(n) \sim_{n\to\infty} (u_\beta + r\, v_\beta)\, h(n)$.*

Claim 4.4 *For $\beta > 0$ and $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbb{N}$ and $C(\varepsilon) > 0$ such that*

$$q_1 * q_2^{r*}(n) \leq C(\varepsilon)\,(1+\varepsilon)^r\, h(n), \qquad n \geq n_0(\varepsilon),\ r \in \mathbb{N}\cup\{0\}\,. \tag{4.9}$$

It remains to use the dominated convergence Theorem to conclude from Claims 4.3 and 4.4 that for $\delta < 1$

$$\lim_{n\to\infty} \frac{1}{h(n)} \sum_{r\geq 1} \delta^r \left[q_1 * q_2^{(r-1)*}\right](n) = u_\beta \sum_{r\geq 1} \delta^r + \delta\, v_\beta \sum_{r\geq 1} r\,\delta^r = \frac{u_\beta\, \delta}{1-\delta} + \frac{v_\beta\, \delta^2}{(1-\delta)^2}\,. \tag{4.10}$$

Combining (4.7) with (4.10) at $\delta = \delta_2(\beta)$ (recall that $\delta_2(\beta) < 1$ by Corollary 3.3) we obtain that

$$\lim_{L\to\infty} \frac{1}{h(L)} \widetilde{Z}^{c}_{L,\beta} = \frac{\bar{K}_\beta}{1-\delta_2(\beta)} + \frac{\widehat{K}_\beta\, \delta_1(\beta)}{(1-\delta_2(\beta))^2}. \tag{4.11}$$

To complete the proof of Theorem 2.1, it remains to take into account trajectories that end with some zero-length stretches. To that aim, we recall (3.8) and we partition Ω_L into subsets whose trajectories are ending with a prescribed number of zero-length stretches, i.e.,

$$\Omega_L = \cup_{k=0}^{L} \left\{ \ell \in \Omega_L : N_\ell \geq k,\ (\ell_i)_{i=1}^{N_\ell - k} \in \Omega^{c}_{L-k},\ \ell_{N_\ell-k+1} = \ell_{N_\ell-k+2} = \cdots = \ell_{N_\ell} = 0 \right\}. \tag{4.12}$$

By using this decomposition, we obtain that

$$\widetilde{Z}_{L,\beta} := Z_{L,\beta}\, e^{-\beta L} = \sum_{k=0}^{L} e^{-\beta k}\, \widetilde{Z}^{c}_{L-k,\beta}\,, \tag{4.13}$$

and then, using (4.11) and dominated convergence we conclude that

$$K_\beta := \lim_{L\to\infty} \frac{1}{h(L)} \widetilde{Z}_{L,\beta} = \frac{1}{1-e^{-\beta}} \Big[\frac{\bar{K}_\beta}{1-\delta_2(\beta)} + \frac{\widehat{K}_\beta\, \delta_1(\beta)}{(1-\delta_2(\beta))^2} \Big], \tag{4.14}$$

which completes the proof of Theorem 2.1, by combining (4.14) with (4.4) and (3.17), and by writing

$$K_\beta \;=\; K^\circ_\beta\, e^{-\beta} \big[(1+e^{-\beta})\, e^{\operatorname{arccosh}(e^{-\beta/2}\cosh(\beta))} - e^{\beta/2}(1-e^{-\beta}) \big]^{-2}, \tag{4.15}$$

where K°_β is computed below in (4.37). □

Proof of Proposition 4.1 Let us now prove Proposition 4.1 subject to Lemma 4.5 and Proposition 4.6 below. Recollecting (3.16), Lemma 4.5 states that the main contribution to the single-bead partition function comes from terms of order $N \approx \sqrt{L}$ in the sum, and Proposition 4.1 provides a sharp estimate of those. The proof of Proposition 4.6 is postponed to Sect. 5, whereas Lemma 4.5 was already stated and proven in [4,Lemma 4.4] so we do not repeat the proof in the present paper.

Recall that N_ℓ is the horizontal extension of a trajectory $\ell \in \Omega^\circ_L$, and $Z^\circ_{L,\beta}(D)$ denotes the partition function restricted to trajectories $\ell \in \Omega^\circ_L$ having property D.

Lemma 4.5 [4,Lemma 4.4] *Let $\beta > \beta_c$, there exists $(a_1, a_2) \in (0,\infty)^2$ such that*

$$\lim_{L\to\infty} \frac{Z^\circ_{L,\beta}(N_\ell \in [a_1,a_2]\sqrt{L})}{Z^\circ_{L,\beta}} = 1\,. \tag{4.16}$$

For $\beta > 0$ and $q > 0$ recall the definitions of $\widetilde{\psi}(q)$ and $C_{\beta,q}$ from (2.5) and (2.9). For conciseness, let us also define

$$\mathcal{V}_{n,k} := \{X\colon X_n = 0,\ A_n = k, X_i > 0, 0 < i < n\}, \quad (n,k) \in (\mathbb{N}_0)^2. \tag{4.17}$$

Proposition 4.6 *Let $[q_1,q_2] \subset (0,\infty)$ and $N \in \mathbb{N}$. We have that for $N \in \mathbb{N}$ such that $qN^2 \in \mathbb{N}$,*

$$\mathbf{P}_\beta(\mathcal{V}_{N,\,qN^2}) \;=\; \frac{C_{\beta,q}}{N^2}\, e^{-N\widetilde{\psi}(q)}(1+o(1))\,, \tag{4.18}$$

uniformly in $q \in [q_1,q_2]$.

Remark 4.7 For the proof of Proposition 4.6 (see Sect. 5), we took inspiration from [15] where a slightly different problem is considered. To be more specific, the authors consider a random walk $(Y_i)_{i\geq 0}$ with a negative drift and a light tail such that the moment generating function $\varphi(t) := E(e^{t\,Y_1})$ satisfies that there exists a $\lambda > 0$ such that $\varphi(\lambda) = 1$, $\varphi^{'}(\lambda) < \infty$ and $\varphi^{''}(\lambda) < \infty$. For such a walk, they provide the assymptotics of the joint law of $\tau := \inf\{i \geq 1\colon\ Y_i \leq 0\}$ and of $A_{\tau-1}$ as the latter becomes large. When applied in our framework, [15,Theorem 1 and 2] prove that for $p > 0$ and $q > p/2$ there exist $C_1, C_2 > 0$ such that

$$\mathbf{P}_\beta(X_N \leq pN, A_N = qN^2,\ X_i > p\quad i,\ 0 < i < N) \underset{N\to\infty}{\sim} \frac{C_1}{N^2} e^{-C_2 N}\,.$$

The case $p = 0$, which is the object of Proposition 4.6 is not covered by [15] though, which is why we provide a complete proof in Sect. 5.

We start the proof of Proposition 4.1 by recalling (3.16) and the equality $\frac{c_\beta}{\Gamma_\beta} = e^{\beta}$, which allow us to write for $L \in \mathbb{N}$,

$$Z^{\circ}_{L,\beta} = 2\, c_\beta\, e^{\beta L}\, \widetilde{Z}^{\circ}_{L,\beta} := 2\, e^{\beta(1+L)} \sum_{N=2}^{1+L/2} (\Gamma_\beta)^N\, \mathbf{P}_\beta(\mathcal{V}_{N,L-N+1})\,. \tag{4.19}$$

Lemma 4.5 guarantees us that it suffices to consider

$$\begin{aligned} Q_{L,\beta} &:= \sum_{N=a_1\sqrt{L}}^{a_2\sqrt{L}} (\Gamma_\beta)^N\, \mathbf{P}_\beta(\mathcal{V}_{N,L-N+1}) \\ &= \sum_{x\in[a_1,a_2]\cap\frac{\mathbb{N}}{\sqrt{L}}} (\Gamma_\beta)^{x\sqrt{L}}\, \mathbf{P}_\beta\big[\mathcal{V}_{x\sqrt{L},\, q_L(x)(x\sqrt{L})^2}\big], \end{aligned} \tag{4.20}$$

with $q_L(x) := \frac{L-x\sqrt{L}+1}{x^2 L}$. We note that $x \in [a_1, a_2]$ yields that for L large enough, $q_L(x) \in [\frac{1}{2a_2^2}, \frac{2}{a_1^2}]$ so that we can apply Proposition 4.6 for every $x \in [a_1, a_2] \cap \frac{\mathbb{N}}{\sqrt{L}}$ in the r.h.s. of (4.20) and obtain

$$Q_{L,\beta} \underset{L\to\infty}{\sim} \sum_{x\in[a_1,a_2]\cap\frac{\mathbb{N}}{\sqrt{L}}} \frac{C_{\beta,q_L(x)}}{x^2\, L}\, e^{x\sqrt{L}\,[\log\Gamma_\beta - \widetilde{\psi}(q_L(x))]}. \tag{4.21}$$

By definition (see (2.5)) $\widetilde{\psi}$ is $\mathcal{C}^2$ on $(0, \infty)$ and therefore, uniformly in $x \in [a_1, a_2]$ we get

$$\widetilde{\psi}(q_L(x)) = \widetilde{\psi}(x^{-2}) - \widetilde{\psi}\,'(x^{-2})\, \frac{1}{x\sqrt{L}} + O(\tfrac{1}{L})\,, \tag{4.22}$$

such that (4.21) becomes

$$Q_{L,\beta} \underset{L\to\infty}{\sim} \frac{1}{L} \sum_{x\in[a_1,a_2]\cap\frac{\mathbb{N}}{\sqrt{L}}} \frac{C_{\beta,q_L(x)} e^{\widetilde{\psi}\,'(x^{-2})}}{x^2}\, e^{\sqrt{L}\,\widetilde{\mathcal{G}}(x)}, \tag{4.23}$$

with $x \in (0, \infty) \mapsto \widetilde{\mathcal{G}}(x) := x \log\Gamma_\beta - x\, \widetilde{\psi}(x^{-2})$ a function already investigated in [4,(1.27)]: which is $\mathcal{C}^2$, negative, has negative second derivative (and therefore is strictly concave on $(0, \infty)$), and reaches its unique maximum at some $a_\beta \in (a_1, a_2)$.

At this stage we pick $R > 0$ and we set $\mathcal{T}_{R,L} := \big[\frac{\lfloor a_\beta\sqrt{L}\rfloor}{\sqrt{L}} - \frac{R}{L^{1/4}}, \frac{\lfloor a_\beta\sqrt{L}\rfloor}{\sqrt{L}} + \frac{R}{L^{1/4}}\big] \cap \frac{\mathbb{N}}{\sqrt{L}}$, and

$$\begin{aligned} \widetilde{Q}^{R,+}_{L,\beta} &:= \sum_{x\in\mathcal{T}_{R,L}} \frac{C_{\beta,q_L(x)} e^{\widetilde{\psi}\,'(x^{-2})}}{x^2}\, e^{\sqrt{L}\,\widetilde{\mathcal{G}}(x)}\,, \\ \widetilde{Q}^{R,-}_{L,\beta} &:= \sum_{x\in[a_1,a_2]\cap\frac{\mathbb{N}}{\sqrt{L}}\setminus\mathcal{T}_{R,L}} \frac{C_{\beta,q_L(x)} e^{\widetilde{\psi}\,'(x^{-2})}}{x^2}\, e^{\sqrt{L}\,\widetilde{\mathcal{G}}(x)}\,. \end{aligned} \tag{4.24}$$

We recall (2.9) and we note that for $\beta > 0$, the function $q \in (0, \infty) \mapsto C_{\beta,q}$ is continuous since $x \in (0, \beta/2) \mapsto \kappa(x)$ is continuous (see Lemma 5.15) as well as $q \mapsto \widetilde{h}^q$ (see (5.6) and (5.7)). Moreover $q_L(x)$ converges to $\frac{1}{a_\beta^2}$ uniformly in $x \in \mathcal{T}_{R,L}$ and therefore

$$\lim_{L\to\infty} C_{\beta,q_L(x)} = C_{\beta,a_\beta^{-2}} \quad \text{uniformly in } x \in \mathcal{T}_{R,L}\,, \tag{4.25}$$

so that we can rewrite

$$\widetilde{Q}_{L,\beta}^{R,+} \underset{L\to\infty}{\sim} \frac{C_{\beta,a_\beta^{-2}}\, e^{\widetilde{\psi}\,'(a_\beta^{-2})}}{a_\beta^2} \sum_{n=-R\,L^{1/4}}^{R\,L^{1/4}} e^{\sqrt{L}\,\widetilde{\mathcal{G}}\big(\frac{\lfloor a_\beta \sqrt{L}\rfloor}{\sqrt{L}}+\frac{n}{\sqrt{L}}\big)}, \tag{4.26}$$

where we have changed the summation indices for computational convenience. Notice that $\widetilde{\mathcal{G}}$ is $\mathcal{C}^2$ on $[a_1, a_2]$ and $\lfloor a_\beta\sqrt{L}\rfloor/\sqrt{L}$ converges to a_β, which is the maximum of $\widetilde{\mathcal{G}}$. Therefore, we may write the following expansion of $\widetilde{\mathcal{G}}$,

$$\widetilde{\mathcal{G}}\Big(\frac{\lfloor a_\beta\sqrt{L}\rfloor}{\sqrt{L}} + \frac{n}{\sqrt{L}}\Big) = \widetilde{\mathcal{G}}(a_\beta) + \frac{1}{2}\ \widetilde{\mathcal{G}}''(a_\beta)\,\frac{n^2}{L} + O\big(\tfrac{n}{L}\big) + o\big(\tfrac{n^2}{L}\big)\,. \tag{4.27}$$

For $\varepsilon > 0$, we deduce an upper bound on the last factor from (4.26),

$$\sum_{n=-R\,L^{1/4}}^{R\,L^{1/4}} e^{\sqrt{L}\,\widetilde{\mathcal{G}}\big(\frac{\lfloor a_\beta\sqrt{L}\rfloor}{\sqrt{L}}+\frac{n}{\sqrt{L}}\big)} \ \le\ e^{\varepsilon R}\, e^{\widetilde{\mathcal{G}}(a_\beta)\sqrt{L}} \sum_{n=-R\,L^{1/4}}^{R\,L^{1/4}} e^{\frac{1}{2}\widetilde{\mathcal{G}}''(a_\beta)(\frac{n}{L^{1/4}})^2}\,, \tag{4.28}$$

for L sufficiently large; and a Riemann sum approximation yields (recall that $\widetilde{\mathcal{G}}''(a_\beta)$ is negative)

$$\sum_{n=-R\,L^{1/4}}^{R\,L^{1/4}} e^{\frac{1}{2}\widetilde{\mathcal{G}}''(a_\beta)(\frac{n}{L^{1/4}})^2} \underset{L\to\infty}{\sim} L^{\frac{1}{4}} \int_{-R}^{R} e^{\widetilde{\mathcal{G}}''(a_\beta)\,\frac{x^2}{2}}\,dx\,. \tag{4.29}$$

Similarly to (4.28), we may write a lower bound on the last factor from (4.26) by changing the sign of ε. Combining both bounds and letting ε go to 0 eventually gives

$$Q_{L,\beta}^{R,+} \underset{L\to\infty}{\sim} \frac{C_{\beta,\,a_\beta^{-2}}\, e^{\widetilde{\psi}\,'(a_\beta^{-2})}}{a_\beta^2}\, L^{\frac{1}{4}}\, e^{\widetilde{\mathcal{G}}(a_\beta)\sqrt{L}} \int_{-R}^{R} e^{\widetilde{\mathcal{G}}''(a_\beta)\,\frac{x^2}{2}}\,dx\,. \tag{4.30}$$

Let us now consider $Q_{L,\beta}^{R,-}$. To that aim, we bound it from above as

$$\widetilde{Q}_{L,\beta}^{R,-} \ \le\ M_{\beta,L} \sum_{x\in[a_1,a_2]\cap\frac{\mathbb{N}}{\sqrt{L}}\setminus\mathcal{T}_{R,L}} e^{\sqrt{L}\,\widetilde{\mathcal{G}}(x)}\,, \tag{4.31}$$

with

$$M_{\beta,L} := \max_{x\in[a_1,a_2]} \frac{C_{\beta,q_L(x)}e^{\widetilde{\psi}\,'(\frac{1}{x^2})}}{x^2}\,. \tag{4.32}$$

The continuity on $(0,\infty)$ of both $q \mapsto C_{\beta,q}$ and $\widetilde{\psi}\,'$ and the fact that, for L large enough, $q_L(x) \in [1/(2a_2^2), 2/a_1^2]$ for $x \in [a_1, a_2]$, guarantees us that there exists a $M > 0$ such that $M_{\beta,L} \le M$ for every $L \ge 1$

Recalling that $\widetilde{\mathcal{G}}$ is $\mathcal{C}^2$, strictly concave and reaches its maximum at a_β there exists $c > 0$ such that $\widetilde{\mathcal{G}}(x) \le \widetilde{\mathcal{G}}(a_\beta) - c\,(x-a_\beta)^2$ for $x \in [a_1, a_2]$ and therefore the sum in the r.h.s. in (4.31) can be bounded above as

$$\sum_{x\in[a_1,a_2]\cap\frac{\mathbb{N}}{\sqrt{L}}\setminus\mathcal{T}_{R,L}} e^{\sqrt{L}\,\widetilde{\mathcal{G}}(x)} \ \le\ 2L^{1/4}e^{\widetilde{\mathcal{G}}(a_\beta)\sqrt{L}}\,\frac{1}{L^{1/4}} \sum_{n=R\,L^{1/4}}^{\infty} e^{-c\,(\frac{n}{L^{1/4}})^2}\,. \tag{4.33}$$

The function $x \mapsto e^{-cx^2}$ being non increasing on $[0, \infty)$, a standard comparison between sum and integral yields that for L large enough and every $R \geq 2$,

$$\frac{1}{L^{1/4}} \sum_{n=R\,L^{1/4}}^{\infty} e^{-c\,(\frac{n}{L^{1/4}})^2} \leq \int_{R-1}^{\infty} e^{-c\,x^2} dx\,. \tag{4.34}$$

It remains to use (4.31–4.34) to conclude that for L large enough and $R \geq 2$,

$$\widetilde{Q}_{L,\beta}^{R,-} \leq 2\,M\,L^{1/4} e^{\widetilde{\mathcal{G}}(a_\beta)\sqrt{L}} \int_{R-1}^{\infty} e^{-c\,x^2} dx\,. \tag{4.35}$$

We recall that $\int_{\mathbb{R}} e^{\widetilde{\mathcal{G}}''(a_\beta)\frac{x^2}{2}} dx = \sqrt{-2\pi/\widetilde{\mathcal{G}}''(a_\beta)}$. Then, we combine (4.23) with (4.24), (4.30) and (4.35) to claim that

$$Q_{L,\beta} \underset{L\to\infty}{\sim} \frac{\sqrt{2\pi}\,C_{\beta,a_\beta^{-2}}\,e^{\widetilde{\psi}'(a_\beta^{-2})}}{a_\beta^2\,\big|\widetilde{\mathcal{G}}''(a_\beta)\big|^{1/2}} \frac{1}{L^{3/4}}\,e^{\widetilde{\mathcal{G}}(a_\beta)\sqrt{L}}\,, \tag{4.36}$$

and it suffices to recall (4.19) to complete the proof of Proposition 4.1 with

$$K_\beta^\circ = 2\,e^\beta\,\frac{\sqrt{2\pi}\,C_{\beta,a_\beta^{-2}}\,e^{\widetilde{\psi}'(a_\beta^{-2})}}{a_\beta^2\,\big|\widetilde{\mathcal{G}}''(a_\beta)\big|^{1/2}}\,. \tag{4.37}$$

□

5 Proof of Proposition 4.6

Proposition 4.6 is a substantial improvement of [4,Prop. 2.5], since this latter proposition only allowed us to bound from below the quantity $\mathbf{P}(\mathcal{V}_{n,qn^2})$ with a polynomial factor $1/n^\gamma$ ($\gamma > 2$) instead of $1/n^2$ in the present Lemma. Let us first recall some results on the large deviation principle satisfied by the sequence of random vectors $(\frac{1}{n}A_{n-1}, X_n)_{n\geq 1}$, which were notably stated in [8,Section 4] in order to study SOS-models. We then provide an outline of the proof of Proposition 4.6, which is divided into 4 steps, corresponding to Sects. 5.1, 5.2, 5.3 and 5.4 respectively.

Change of Measure

Let $X := (X_i)_{i\in\mathbb{N}}$ be a random walk with law $\mathbf{P}_{\beta,0}$, i.e. starting from the origin and whose increments are i.i.d. with law $\mathbf{P}_\beta$ (recall (2.1)). For $|h| < \beta/2$, recall the definitions of $\mathcal{L}(h)$ and $\widetilde{\mathbf{P}}_h$ in (2.2), and (2.7). Recall also the definition of $A_N(\cdot)$ in (2.4).

For a trajectory $x = (x_i)_{i=0}^n \in \mathbb{Z}^{n+1}$, $x_0 = 0$, define $\Lambda_n(x) := \big(\frac{1}{n}A_{n-1}(x), x_n\big)$. Large deviations estimates for the random vector $\Lambda_n(X)$ are displayed in [8]. Typically, one is interested in the probability of events as

$$\{\tfrac{1}{n}\Lambda_n(X) = (q,p)\} \quad \text{with} \quad (q,p) \in \mathbb{R}^* \times \mathbb{R}\,,$$

(where $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$), which requires to introduce tilted probability laws of the form

$$\frac{\mathrm{d}\mathbf{P}_{n,\boldsymbol{h}}}{\mathrm{d}\mathbf{P}_{\beta,0}}(x) := e^{\boldsymbol{h}\cdot\Lambda_n(x) - \mathcal{L}_{\Lambda_n}(\boldsymbol{h})} \quad \text{with} \quad \mathcal{L}_{\Lambda_n}(\boldsymbol{h}) := \log \mathbf{E}_{\beta,0}[e^{\boldsymbol{h}\cdot\Lambda_n(X)}]. \tag{5.1}$$

for $x=(x_i)_{i=0}^n \in \mathbb{Z}^{n+1}$, $x_0=0$, where $\boldsymbol{h} := (h_0,h_1) \in \mathcal{D}_{\beta,n}$ with

$$\mathcal{D}_{\beta,n} := \big\{(h_0,h_1)\in\mathbb{R}^2\,;\ |h_1|<\beta/2,\ |(1-\tfrac{1}{n})h_0+h_1|<\beta/2\big\}.$$

For $(q,p)\in\mathbb{R}^*\times\mathbb{R}$, the fact that $\nabla[\frac{1}{n}\mathcal{L}_{\Lambda_n}]$ is a $\mathcal{C}^1$ diffeomorphism from $\mathcal{D}_{\beta,n}$ to $\mathbb{R}^2$ (see [4,Lemma 5.4]), allows us to choose $\boldsymbol{h} := \boldsymbol{h}_n(q,p)$ in (5.1) with $\boldsymbol{h}_n(q,p)$ the unique solution (in $\boldsymbol{h}$) of the equation

$$\mathbf{E}_{n,\boldsymbol{h}}\Big[\frac{1}{n}\Lambda_n(X)\Big] = \nabla\Big[\frac{1}{n}\mathcal{L}_{\Lambda_n}\Big](\boldsymbol{h}) = (q,p). \tag{5.2}$$

Under $\mathbf{P}_{n,\boldsymbol{h}_n(q,p)}$ the event $\{\frac{1}{n}\Lambda_n(X)=(q,p)\}$ becomes typical and can be sharply estimated.

In the present context though we only consider events where the random walk returns to the origin after n steps, i.e., $\{\frac{1}{n}\Lambda_n(X)=(q,0)\}$ with $p=0$. Moreover, for straightforward symmetry reasons (stated e.g. in [4,Remark 5.5]) we have

$$\boldsymbol{h}_n(q,0) = \big(h_n^q, -h_n^q\,(\tfrac{1}{2}-\tfrac{1}{2n})\big), \quad q\in\mathbb{R}$$

where h_n^q is the unique solution in h of the equation $\mathcal{G}_n'(h)=q$, with

$$\begin{aligned}\mathcal{G}_n(h) &:= \frac{1}{n}\mathcal{L}_{\Lambda_n}\big[h, -\tfrac{h}{2}(1-\tfrac{1}{n})\big] \quad \text{for} \quad h\in\big(-\tfrac{n\beta}{n-1},\tfrac{n\beta}{n-1}\big)\\ &= \tfrac{1}{n}\sum_{i=1}^n \mathcal{L}\big[\tfrac{h}{2}(1-\tfrac{2i-1}{n})\big].\end{aligned} \tag{5.3}$$

It is claimed in Lemma 5.2 below that $h\mapsto\mathcal{G}_n'(h)$ is a $\mathcal{C}^1$ diffeomorphism from $(-\frac{n\beta}{n-1},\frac{n\beta}{n-1})$ to $\mathbb{R}$ which justifies the existence and uniqueness of h_n^q. As a consequence, instead of those tilted probability laws in (5.1), we shall rather use the probability laws $\mathbf{P}_{n,h}$ that depend on the sole parameter $h\in(-\frac{n\beta}{n-1},\frac{n\beta}{n-1})$, i.e. for $x=(x_i)_{i=0}^n\in\mathbb{Z}^{n+1}$, $x_0=0$,

$$\frac{\mathrm{d}\mathbf{P}_{n,h}}{\mathrm{d}\mathbf{P}_{\beta,0}}(x) = e^{\psi_{n,h}(A_{n-1}(x),x_n)}, \quad \text{with} \quad \psi_{n,h}(a,s) := h\,\frac{a}{n} - \frac{h}{2}\Big(1-\frac{1}{n}\Big)s - n\,\mathcal{G}_n(h)\,, \quad s,a\in\mathbb{Z}\,. \tag{5.4}$$

Let us point out that, under $\mathbf{P}_{n,h}$, the increments $(X_i-X_{i-1})_{i=1}^n$ are independent and X_i-X_{i-1} follows the law $\widetilde{\mathbf{P}}$ with parameter $\frac{h}{2}(1-\frac{2i-1}{n})$ (recall (2.7)).

It remains to define the continuous counterpart of $\mathcal{L}_{\Lambda_n}$.

$$\mathcal{D}_\beta := \Big\{(h_0,h_1)\in\mathbb{R}^2\,;\ |h_1|<\beta/2,\ |h_0+h_1|<\beta/2\Big\}, \tag{5.5}$$

and for every $\boldsymbol{h}=(h_0,h_1)\in\mathcal{D}_\beta$,

$$\mathcal{L}_\Lambda(\boldsymbol{h}) := \int_0^1 \mathcal{L}(h_0x+h_1)\mathrm{d}x. \tag{5.6}$$

As stated in [8] and [4,Lemma 5.3], $\nabla\mathcal{L}_\Lambda(\boldsymbol{h})$ which can be written as

$$\begin{aligned}\nabla\mathcal{L}_\Lambda(\boldsymbol{h}) &= (\partial_{h_0}\mathcal{L}_\Lambda, \partial_{h_1}\mathcal{L}_\Lambda)(\boldsymbol{h})\\ &= \Big(\int_0^1 x\mathcal{L}'(xh_0+h_1)\mathrm{d}x, \int_0^1 \mathcal{L}'(xh_0+h_1)\mathrm{d}x\Big),\end{aligned} \tag{5.7}$$

is a $\mathcal{C}^1$ diffeomorphism from $\mathcal{D}_\beta$ to $\mathbb{R}^2$. Thus, for $(q,p)\in\mathbb{R}^2$ we let $\widetilde{\boldsymbol{h}}(q,p)$ be the unique solution in $\boldsymbol{h}\in\mathcal{D}_\beta$ of the equation $\nabla\mathcal{L}_\Lambda(\boldsymbol{h})=(q,p)$. As mentioned above for the discrete case we only need the case $p = 0$, for which the fact that $\mathcal{L}'$ is an odd increasing function combined with (5.7) ensures us that

$$\widetilde{\boldsymbol{h}}(q, 0) = \Big(\widetilde{h}^q, -\frac{\widetilde{h}^q}{2}\Big), \quad q \in \mathbb{R}, \tag{5.8}$$

where $\widetilde{h}^q$ is the unique solution in h of the equation $\mathcal{G}'(h) = q$, with

$$\begin{aligned}\mathcal{G}(h) &:= \mathcal{L}_\Lambda\big(h, -\tfrac{h}{2}\big) \quad \text{for} \quad h \in (-\beta, \beta) \\ &= \int_0^1 \mathcal{L}(h(\tfrac{1}{2} - x))\mathrm{d}x\,. \end{aligned} \tag{5.9}$$

With Lemma 5.3 below, we claim that $h \mapsto \mathcal{G}'(h)$ is a $\mathcal{C}^1$ diffeomorphism from $(-\beta, \beta)$ to $\mathbb{R}$, which justifies the existence and uniqueness of $\widetilde{h}^q$.

Outline of the Proof of Proposition 4.6 With Step 1 below, we bound from above the difference between the finite size exponential decay rate of $\mathbf{P}_\beta(\mathcal{V}_{n,qn^2})$ and its limit as n tends to ∞. In Step 2, we divide $\mathbf{P}_\beta(\mathcal{V}_{N,qN^2})$ into a main term $M_{N,q}$ and an error term $E_{N,q}$. The main term is obtained by adding to the definition of $\mathcal{V}_{N,qN^2}$ some constraints concerning the possible values taken by $X_{a_N}, A_{a_N}, X_{N-a_N}$ and A_{N-a_N-1} for an ad hoc sequence $(a_N)_{N\geq 1}$ satisfying both $a_N = o(N)$ and $\lim_{n\to\infty} a_N = \infty$. The $E_{N,q}$ term is bounded above in Step 3, while in Step 4 we provide a sharp estimate of $M_{N,q}$, and we conclude the proof by computing the pre-factors in the estimate of $\mathbf{P}_\beta(\mathcal{V}_{N,qN^2})$.

5.1 Step 1

The aim of this step is to prove the following Proposition, which is a strong improvement of [4,Proposition 2.3] since we bound from above the gap between discrete quantities and their continuous counterparts by n^{-2} instead of n^{-1}. As mentioned in Remark 4.7 above, our proof is close in spirit to that of [15,Theorem 2], in particular for Proposition 5.4 below. Recall the definitions of $\psi_{n,h}$ and $\widetilde{\psi}$ from (5.4) and (2.5) respectively, and notice that $q \mapsto \frac{1}{n}\psi_{n,h_n^q}(qn^2, 0)$ (resp. $q \mapsto \widetilde{\psi}(q)$) is the Legendre transform of $\mathcal{G}_n$ (resp. $\mathcal{G}$).

Proposition 5.1 *For $[q_1, q_2] \subset (0, \infty)$, there exists constants $C > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and $q \in [q_1, q_2] \cap \frac{\mathbb{N}}{n^2}$,*

$$\left|\frac{1}{n}\psi_{n,h_n^q}(qn^2, 0) - \widetilde{\psi}(q)\right| \leq \frac{C}{n^2}, \tag{5.10}$$

and

$$\left|h_n^q - \widetilde{h}^q\right| \leq \frac{C}{n^2}. \tag{5.11}$$

We recall (5.3) and (5.9) where the definitions of $\mathcal{G}_n$ and $\mathcal{G}$ are displayed respectively. Let us first claim two lemmata which state that $\mathcal{G}_n'$ and $\mathcal{G}'$ are $\mathcal{C}^1$-diffeomorphisms.

Lemma 5.2 *For every $n \geq 2$, the function $\mathcal{G}_n$ is $\mathcal{C}^\infty$, strictly convex, even and satisfies $\mathcal{G}_n'(h) \to +\infty$ as $h \nearrow \frac{n\beta}{n-1}$. Moreover, there exists $R > 0$ such that $(\mathcal{G}_n)''(h) \geq R$ for every $n \geq 2$ and $h \in (-\frac{n\beta}{n-1}, \frac{n\beta}{n-1})$. As a consequence, $(\mathcal{G}_n)'$ is an increasing $\mathcal{C}^1$ diffeomorphism from $(-\frac{n\beta}{n-1}, \frac{n\beta}{n-1})$ to $\mathbb{R}$ and $(\mathcal{G}_n)'(0) = 0$.*

Lemma 5.3 *The function $\mathcal{G}$ is $\mathcal{C}^\infty$, strictly convex, even and satisfies $\mathcal{G}'(h) \to +\infty$ as $h \nearrow \beta$. Moreover, there exists an $R > 0$ such that $(\mathcal{G})''(h) \geq R$ for $h \in (-\beta, \beta)$. As a consequence, $(\mathcal{G})'$ is an increasing $\mathcal{C}^1$ diffeomorphism from $(-\beta, \beta)$ to $\mathbb{R}$ and $(\mathcal{G})'(0) = 0$.*

Those results are straightforward consequences of (2.2), so we do not detail them any further. Let us then recall that h_n^q is the unique solution in h of $(\mathcal{G}_n)'(h) = q$ and that $\widetilde{h}^q$ is the unique solution in h of $\mathcal{G}'(h) = q$. At this stage, we state a key result which substantially improves [4,Lemma 5.1] and [8,(4.7)], giving an upper bound of order n^{-2} instead of n^{-1}. Its proof is postponed to Appendix A.2.

Proposition 5.4 *For every $K \in (0, \beta)$, there exist a $C_K > 0$ and a $n_K \in \mathbb{N}$ such that for $j \in \{0, 1\}$*

$$\sup_{h\in[-\beta+K,\beta-K]} \left|\mathcal{G}_n^{(j)}(h) - \mathcal{G}^{(j)}(h)\right| \leq \frac{C_K}{n^2}, \quad \forall n \geq n_K. \tag{5.12}$$

With the help of Lemma 5.3 we can state the following corollary of Proposition 5.4.

Corollary 5.5 *For every $M > 0$, there exists a $K > 0$ such that $\mathcal{G}'(\beta - K) \geq 2M$ and $\mathcal{G}'(K) \leq \frac{1}{2M}$; and there exists an $n_0 \in \mathbb{N}$ such that $\mathcal{G}_n'(\beta - K) \geq M$ and $\mathcal{G}_n'(K) \leq \frac{1}{M}$ for every $n \geq n_0$.*

Remark 5.6 A straightforward consequence of Corollary 5.5 is that for $[q_1, q_2] \subset (0, \infty)$ there exists a $K > 0$ and an $n_0 \in \mathbb{N}$ such that for every $q \in [q_1, q_2]$ and every $n \geq n_0$ we have $h_n^q, \widetilde{h}^q \in [K, \beta - K]$. Moreover, since $\mathcal{L}$ is $\mathcal{C}^\infty$ on $(-\beta/2, \beta/2)$ we deduce for $j \in \mathbb{N} \cup \{0\}$ and $0 < q_1 < q_2 < +\infty$,

$$\sup\left\{\left|\mathcal{L}^{(j)}\big(h_n^q(\tfrac{1}{2} - x)\big)\right|;\ x \in [0, 1],\ q \in [q_1, q_2],\ n \in \mathbb{N}\right\} < +\infty.$$

Proof of Proposition 5.1 We now have all the required tools in hand to prove Proposition 5.1. By using Lemma 5.3, Proposition 5.4 and Remark 5.6, we can state that there exist $K > 0$, $C > 0$ and $R > 0$ such that for n large enough and $q \in [q_1, q_2] \cap \mathbb{N}/n^2$ we have $h_n^q, \widetilde{h}^q \in [K, \beta - K]$, so

$$\begin{aligned}\left|h_n^q - \widetilde{h}^q\right| &\leq \frac{1}{R}\left|\int_{h_n^q}^{\widetilde{h}^q} \mathcal{G}''(x)\,dx\right| = \frac{1}{R}\left|\mathcal{G}'(h_n^q) - \mathcal{G}'(\widetilde{h}^q)\right| \\ &= \frac{1}{R}\left|\mathcal{G}'(h_n^q) - \mathcal{G}_n'(h_n^q)\right| \leq \frac{C}{R\,n^2},\end{aligned} \tag{5.13}$$

where we have used that $q = \mathcal{G}'(\widetilde{h}^q) = \mathcal{G}_n'(h_n^q)$ for the second equality in (5.13). The proof of (5.11) is therefore completed.

It remains to prove (5.10). To that aim we write

$$\left|h_n^q\, q - \mathcal{G}_n(h_n^q) - \widetilde{h}^q\, q + \mathcal{G}(\widetilde{h}^q)\right| \leq U_{n,q} + q_2\left|h_n^q - \widetilde{h}^q\right|, \tag{5.14}$$

where $U_{n,q} := |\mathcal{G}_n(h_n^q) - \mathcal{G}(\widetilde{h}^q)|$. Proposition 5.4 also tells us that there exists a $C > 0$ such that for n large enough and for every $x \in [K, \beta - K]$ we have $|\mathcal{G}_n(x) - \mathcal{G}(x)| \leq C/n^2$. Thus, since $\mathcal{G}$ is $\mathcal{C}^1$ and recalling Remark 5.6 we can write

$$\begin{aligned}U_{n,q} &\leq \left|\mathcal{G}_n(h_n^q) - \mathcal{G}(h_n^q)\right| + \left|\mathcal{G}(h_n^q) - \mathcal{G}(\widetilde{h}_0^q)\right| \\ &\leq \frac{C}{n^2} + \sup\left\{|\mathcal{G}'(x)|, x \in [K, \beta - K]\right\}\left|h_n^q - \widetilde{h}^q\right|.\end{aligned} \tag{5.15}$$

At this stage, (5.10) is obtained by combining (5.14) with (5.11) and (5.15). This completes the proof of Proposition 5.1. □

5.2 Step 2

We recall (4.17) and (5.4) and we pick a sequence $a_N \in \mathbb{N}$, $N \in \mathbb{N}$ such that $\log N \ll a_N \ll \sqrt{N}$. We define the two boxes

$$\mathcal{C}_N := \big[\mathbf{E}_{N,h_N^q}(X_{a_N}) - (a_N)^{3/4}, \mathbf{E}_{N,h_N^q}(X_{a_N}) + (a_N)^{3/4}\big],$$
$$\mathcal{D}_N := \big[\mathbf{E}_{N,h_N^q}(A_{a_N}) - (a_N)^{7/4}, \mathbf{E}_{N,h_N^q}(A_{a_N}) + (a_N)^{7/4}\big], \tag{5.16}$$

and rewrite $\mathbf{P}_\beta(\mathcal{V}_{N,qN^2}) = M_{N,q} + E_{N,q}$ where

$$M_{N,q} := \mathbf{P}_\beta\big[\mathcal{V}_{N,qN^2} \cap \big\{X_{a_N} \in \mathcal{C}_N, A_{a_N} \in \mathcal{D}_N\big\} \cap \big\{X_{N-a_N} \in \mathcal{C}_N, A_N - A_{N-a_N-1} \in \mathcal{D}_N\big\}\big]. \tag{5.17}$$

From now on, $M_{N,q}$ is referred to as the main term and $E_{N,q}$ as the error term. The proof of Proposition 4.6 is a straightforward consequence of Lemmata 5.7–5.8 displayed below, which are proven in Steps 3 and 4, respectively.

Let us start with the following Lemma, which allows us to control the error therm uniformly in q belonging to any compact set of $(0, \infty)$.

Lemma 5.7 *For* $[q_1, q_2] \subset (0, \infty)$*, there exists* $\varepsilon : \mathbb{N} \mapsto \mathbb{R}^+$ *such that* $\lim_{N\to\infty} \varepsilon(N) = 0$ *and for every* $N \in \mathbb{N}$ *and* $q \in [q_1, q_2] \cap \frac{1}{N^2}$

$$E_{N,q} \le \frac{\varepsilon(N)}{N^2} e^{-N\widetilde{\psi}(q)}. \tag{5.18}$$

With the next Lemma, we estimate the main therm uniformly in q belonging to any compact set of $(0, \infty)$. Recall (2.8) and (2.6) for the definitions of κ and ϑ.

Lemma 5.8 *For* $\beta > 0$ *and* $[q_1, q_2] \subset (0, \infty)$*,*

$$M_{N,q} = \kappa\big(\tfrac{\widetilde{h}^q}{2}\big)^2 \frac{\big(\vartheta(\widetilde{h}^q)\big)^{-\frac{1}{2}}}{2\pi N^2} e^{-N\widetilde{\psi}(q)} (1 + o(1)), \tag{5.19}$$

where $o(1)$ *is a function that converges to* 0 *as* $N \to \infty$ *uniformly in* $q \in [q_1, q_2] \cap \frac{\mathbb{N}}{N^2}$.

5.3 Step 3: Proof of Lemma 5.7

Before starting the proof, let us make a quick remark about the time-reversibility of the random walk X of law $\mathbf{P}_{n,h}$.

Remark 5.9 (Time-reversal property) Recall the definition of $\widetilde{\mathbf{P}}_h$ from (2.7). For $h \in (-\beta/2, \beta/2)$, one can easily check that if Z is a random variable of law $\widetilde{\mathbf{P}}_{-h}$ then $-Z$ has law $\widetilde{\mathbf{P}}_h$. Moreover, as explained below (5.4), if the n-step random walk $X := (X_i)_{i=0}^n$ has law $\mathbf{P}_{n,h}$, then its increments $(X_i - X_{i-1})_{i=1}^n$ are independent and for $i \in \{1, \dots, n\}$ the law of $X_i - X_{i-1}$ is $\widetilde{\mathbf{P}}$ with parameter $\frac{h}{2}(1 - \frac{2i-1}{n})$. A first consequence is that X is time-reversible, i.e.,

$$(X_i)_{i=0}^n \underset{\text{Law}}{=} (X_{n-i} - X_n)_{i=0}^n. \tag{5.20}$$

A second consequence is that, under $\mathbf{P}_{n,h}$, the random walk $(X_i)_{i=0}^n$ is an inhomogeneous Markov chain which, for $j \in \{1, \dots, N-1\}$, $y \in \mathbb{Z}$ and $\mathcal{O} \subset \mathbb{Z}^{n-j-1}$ satisfies that

$$\mathbf{P}_{n,h}\big((X_{j+i})_{i=1}^{n-j-1} \in \mathcal{O},\ X_n = 0 \mid X_j = y\big) = \mathbf{P}_{n,h}\big((X_{n-j-i})_{i=1}^{n-j-1} \in \mathcal{O},\ X_{n-j} = y\big). \tag{5.21}$$

Note finally that the case $h = 0$ corresponds to the random walk X with i.i.d. increments of law $\mathbf{P}_\beta$.

A straightforward application of (5.20) with $n = N$ and $h = 0$ allows us to bound the error term from above as

$$E_{N,q} \le 2\,\mathbf{P}_\beta\big(\mathcal{V}_{N,qN^2} \cap \big\{X_{a_N} \notin \mathcal{C}_N\}\big) + 2\,\mathbf{P}_\beta\big(\mathcal{V}_{N,qN^2} \cap \big\{A_{a_N} \notin \mathcal{D}_N\}\big). \tag{5.22}$$

Using (5.4) we obtain that for $\mathcal{B} = \{X_{a_N} \notin \mathcal{C}_N\}$ or $\mathcal{B} = \{A_{a_N} \notin \mathcal{D}_N\}$

$$\mathbf{P}_\beta\big(\mathcal{V}_{N,qN^2} \cap \mathcal{B}\big) \le e^{-\psi_{N,h_N^q}(qN^2,0)}\, \mathbf{P}_{N,h_N^q}(\mathcal{V}_{N,qN^2} \cap \mathcal{B}). \tag{5.23}$$

Note that, the first inequality in Proposition 5.1 allows us to replace $\psi_{N,h_N^q}(qN^2,0)$ with $N\widetilde{\psi}(q)$ in the exponential of the r.h.s. in (5.23), at the cost of an at most constant factor. Therefore, the proof of Lemma 5.7 is completed by the following Claim.

Claim 5.10 *For* $[q_1, q_2] \subset (0,\infty)$, *there exists* $\varepsilon : \mathbb{N} \mapsto \mathbb{R}^+$ *such that* $\lim_{N\to\infty} \varepsilon(N) = 0$ *and for every* $N \in \mathbb{N}$ *and* $q \in [q_1, q_2] \cap \frac{\mathbb{N}}{N^2}$

$$\mathbf{P}_{N,h_N^q}(\mathcal{V}_{N,qN^2} \cap \{X_{a_N} \notin \mathcal{C}_N\}) + \mathbf{P}_{N,h_N^q}(\mathcal{V}_{N,qN^2} \cap \{A_{a_N} \notin \mathcal{D}_N\}) \le \frac{\varepsilon(N)}{N^2}. \tag{5.24}$$

Proof Let us prove that (5.24) holds true for $R_{N,q} := \mathbf{P}_{N,h_N^q}(\mathcal{V}_{N,qN^2} \cap \{X_{a_N} \notin \mathcal{C}_N\})$ and for $S_{N,q} := \mathbf{P}_{N,h_N^q}(\mathcal{V}_{N,qN^2} \cap \{A_{a_N} \notin \mathcal{D}_N\})$.

We set $\{X_{[j,k]} > 0\} := \{X_i > 0,\ j \le i \le k\}$ for $j \le k \in \mathbb{N}$. We develop $R_{N,q}$ depending on the values y and z taken by X_{a_N} and A_{a_N} respectively. Then we use Markov property at time a_N, combined with the time reversal property (5.21) with $n = N$, $h = h_N^q$, $j = a_N$ and

$$\mathcal{O} = \Big\{x \in \mathbb{N}^{N-j-1} \colon \sum_{i=1}^{N-j-1} x_i = qN^2 - z\Big\},$$

on the time interval $[a_N, N]$ to obtain

$$\begin{aligned} R_{N,q} = \sum_{y\in\mathbb{N}\setminus\mathcal{C}_N} \sum_{z\in\mathbb{N}} &\mathbf{P}_{N,h_N^q}(X_{a_N} = y,\ A_{a_N} = z,\ X_{[1,a_N]} > 0) \\ &\times \mathbf{P}_{N,h_N^q}(X_{[1,N-a_N]} > 0, X_{N-a_N} = y, A_{N-a_N-1} = qN^2 - z). \end{aligned} \tag{5.25}$$

Let $R^1_{N,q} := R_{N,q}(|y| \le N/a_N, |z| \le N^2/a_N)$, $R^2_{N,q} := R_{N,q}(|y| > N/a_N)$ and $R^3_{N,q} := R_{N,q}(|z| > N^2/a_N)$, where $R_{N,q}(A)$ denotes the sum from (5.25) restricted to terms satisfying the condition A; so that

$$R_{N,q} \ \le\ R^1_{N,q} + R^2_{N,q} + R^3_{N,q}. \tag{5.26}$$

Let us prove the upper bound (5.24) for all three terms in the r.h.s., starting with $R^1_{N,q}$.

Lemma 5.11 *For* $[q_1, q_2] \subset (0,\infty)$, *there exists a* $C > 0$ *such that for every* $N \ge 1$ *and* $q \in [q_1, q_2] \cap \frac{\mathbb{N}}{N^2}$ *and* $(y,z) \in \mathbb{Z}^2$ *with* $|y| \le N/a_N$, $|z| \le N^2/a_N$,

$$\mathbf{P}_{N,h_N^q}(X_{N-a_N} = y, A_{N-a_N-1} = qN^2 - z) \le \frac{C}{N^2}. \tag{5.27}$$

We prove this Lemma afterwards. Plugging this in the development (5.25) and dropping the condition $X_{[1,a_N]} > 0$, we obtain

$$R^1_{N,q} \leq \frac{C}{N^2}\mathbf{P}_{N,h_N^q}(X_{a_N} \notin \mathcal{C}_N)\,. \tag{5.28}$$

Recalling (5.4) and the subsequent remark, a straightforward computation gives us that

$$\mathrm{Var}_{N,h_N^q}\big(X_{a_N}\big) = \sum_{i=1}^{a_N} \mathcal{L}''\Big(\tfrac{h_N^q}{2}(1-\tfrac{2i-1}{N})\Big)\,, \tag{5.29}$$

where we used, for U distributed as $\widetilde{\mathbf{P}}_\delta$ and $|\delta| < \beta/2$, $\mathrm{Var}(U) = \mathcal{L}''(\delta)$ (recall (2.7)). By using Tchebychev inequality we obtain

$$\begin{aligned}
\mathbf{P}_{N,h_N^q}(X_{a_N} \notin \mathcal{C}_N) &= \mathbf{P}_{N,h_N^q}\big(|X_{a_N} - \mathbf{E}_{N,h_N^q}(X_{a_N})| > (a_N)^{3/4}\big)\\
&\leq \frac{1}{a_N^{3/2}}\mathrm{Var}_{N,h_N^q}\big(X_{a_N}\big)\\
&\leq \frac{1}{a_N^{1/2}} \sup_{x\in[0,1]} \Big|\mathcal{L}''\Big(h_N^q(\tfrac{1}{2}-x)\Big)\Big|\\
&\leq \frac{(\text{const.})}{\sqrt{a_N}}\,,
\end{aligned} \tag{5.30}$$

where the last inequality is a consequence of Remark 5.6; in particular the constant is uniform in $N \in \mathbb{N}$ and $q \in [q_1, q_2] \cap \frac{\mathbb{N}}{N^2}$. It remains to combine (5.28) and (5.30) to complete the proof for $R^1_{N,q}$.

Regarding $R^2_{N,q}$, let $(U_i)_{i\geq 1}$ be the increments of the process X. Recalling (5.4) and applying a Chernov inequality for some $\lambda > 0$ small, we have

$$\begin{aligned}
&\sum_{y>N/a_N} \mathbf{P}_{N,h_N^q}(X_{a_N} = y, A_{a_N} = z, X_{[1,a_N]} > 0)\\
&\leq\ \mathbf{P}_{N,h_N^q}(X_{a_N} > N/a_N)\\
&\leq\ \mathbf{E}_{\beta,0}\left[\exp\left(\lambda(X_{a_N} - N/a_N) + \sum_{i=1}^{a_N} U_i\Big(\frac{h_N^q}{2}\Big(1-\frac{2i-1}{N}\Big)\Big) - \mathcal{L}\Big(\frac{h_N^q}{2}\Big(1-\frac{2i-1}{N}\Big)\Big)\right)\right]\\
&\leq\ e^{-\lambda N/a_N} e^{c\,a_N}\ \leq\ C\,e^{-\frac{\lambda N}{2a_N}}\,,
\end{aligned} \tag{5.31}$$

for some $c > 0$, $C > 0$, which can be taken uniformly in $q \in [q_1, q_2]$. The same upper bound holds for the sum over $y < -N/a_N$, hence

$$\begin{aligned}
R^2_{N,q}\ &\leq\ \sum_{z\in\mathbb{Z}} \mathbf{P}_{N,h_N^q}(A_{N-a_N-1} = qN^2 - z)\\
&\qquad\times \sum_{|y|>N/a_N} \mathbf{P}_{N,h_N^q}(X_{a_N} = y, A_{a_N} = z, X_{[1,a_N]} > 0)\\
&\leq\ 2\,C\,e^{-\frac{\lambda N}{2a_N}}\,.
\end{aligned} \tag{5.32}$$

The last term $R^3_{N,q}$ can be handled similarly, by noticing that $A_{a_N} > N^2/a_N$ implies $X_i > N^2/a_N^2$ for some $1 \le i \le a_N$, thereby

$$\sum_{z>N^2/a_N} \mathbf{P}_{N,h_N^q}(X_{a_N} = y, A_{a_N} = z, X_{[1,a_N]} > 0)$$
$$\le \mathbf{P}_{N,h_N^q}(A_{a_N} > N^2/a_N) \le \sum_{i=1}^{a_N} \mathbf{P}_{N,h_N^q}(X_i > N^2/a_N^2) \le C\, a_N e^{-\frac{\lambda N^2}{2a_N^2}},$$

where we reproduced (5.31). Similarly to (5.32), we obtain an upper bound on $R^3_{N,q}$, and recollecting (5.26) with (5.28), (5.30) and (5.32), this finally proves (5.24) for $R_{N,q}$.

Let us now consider $S_{N,q}$, which we develop similarly to (5.25). Notice that the term $S_{N,q}(y \notin \mathcal{C}_N)$ is already bounded from above by $R_{N,q}$, and that $S_{N,q}(|z| > N^2/a_N)$ follows the same upper bound as $R^3_{N,q}$ (we do not replicate the proof), thus we only have to prove (5.24) for $S^1_{N,q} := S_{N,q}(y \in \mathcal{C}_N, |z| \le N^2/a_N)$ to complete the proof of Claim 5.10. Recalling Lemma 5.11, we have

$$S^1_{N,q} \le \frac{C}{N^2}\mathbf{P}_{N,h_N^q}(A_{a_N} \notin \mathcal{D}_N). \tag{5.33}$$

Let $(U_i)_{i\ge 1}$ denote the increments of the process X, and recall $A_{a_N} = \sum_{i=1}^{a_N} X_i$; in particular, we may rewrite $A_{a_N} = \sum_{i=1}^{a_N}(a_N + 1 - i)U_i$. Similarly to (5.29), a direct computation gives

$$\mathrm{Var}_{N,h_N^q}\big(A_{a_N}\big) = \sum_{i=1}^{a_N}(a_N + 1 - i)^2 \mathcal{L}''\Big(\tfrac{h_N^q}{2}(1 - \tfrac{2i-1}{N})\Big),$$

and using Tchebychev inequality, we obtain

$$\begin{aligned}\mathbf{P}_{N,h_N^q}(A_{a_N} \notin \mathcal{D}_N) &\le \frac{1}{a_N^{7/2}}\mathrm{Var}_{N,h_N^q}\big(A_{a_N}\big)\\ &\le \frac{1}{a_N^{1/2}} \sup_{x\in[0,1]}\Big|\mathcal{L}''\Big(h_N^q(\tfrac12 - x)\Big)\Big|\\ &\le \frac{\text{(const.)}}{\sqrt{a_N}},\end{aligned} \tag{5.34}$$

for some constant uniform in $N \in \mathbb{N}$ and $q \in [q_1, q_2] \cap \frac{\mathbb{N}}{N^2}$, where the last inequality is a consequence of Remark 5.6. Combining this with (5.33), this concludes the proof of the Claim. □

Proof of Lemma 5.11 To lighten upcoming formulae in this proof, let us write $N_1 := N - a_N$. Recall (5.4), and that $\mathbf{P}_{\beta,y}$ denotes the law of the random walk starting from $y \in \mathbb{Z}$ with increments distributed as $\mathbf{P}_\beta$. Summing over possible values for (X_N, A_{N-1}) and using Markov property, we write

$$\begin{aligned}&\mathbf{P}_{N,h_N^q}(X_{N_1} = y, A_{N_1-1} = qN^2 - z)\\ &\quad= \sum_{(u,v)\in\mathbb{Z}^2} \mathbf{P}_{N,h_N^q}(X_{N_1} = y, A_{N_1-1} = qN^2 - z, X_N = u, A_{N-1} = v)\\ &\quad= \sum_{(u,v)\in\mathbb{Z}^2} e^{\psi_{N,h_N^q}(v,u)}\mathbf{P}_{\beta,0}(X_{N_1} = y, A_{N_1-1} = qN^2 - z, X_N = u, A_{N-1} = v)\end{aligned}$$

$$= \sum_{(u,v)\in\mathbb{Z}^2} e^{\psi_{N,h_N^q}(v,u)} \mathbf{P}_{\beta,0}(X_{N_1} = y, A_{N_1-1} = qN^2 - z)\mathbf{P}_{\beta,y}(X_{a_N} = u, A_{a_N-1} = v - qN^2 + z - y) \quad = \mathbf{P}_{N_1,h_{N_1}^q}(X_{N_1} = y, A_{N_1-1} = qN^2 - z) \times \sum_{(u,v)\in\mathbb{Z}^2} \exp\Big(\psi_{N,h_N^q}(v,u) - \psi_{N_1,h_{N_1}^q}(qN^2 - z, y)\Big) \mathbf{P}_{\beta,y}(X_{a_N} = u, A_{a_N-1} = v - qN^2 + z - y), \tag{5.35}$$

where we obtained the third equality by splitting the walk at time N_1, requiring the second bit to start from height y and to cover an area $v - A_{N_1-1} - X_{N_1} = v - qN^2 + z - y$ (recall the definition of $A_n(X)$, $n \in \mathbb{N}$ from (2.4)). A straightforward consequence of Proposition 6.1 and Remark 6.2 from Appendix A.1 is that the first factor is uniformly controlled by $\frac{C}{N^2}$ for some uniform $C > 0$, so it remains to prove that the second factor is uniformly bounded. We notice

$$\mathbf{P}_{\beta,y}(X_{a_N} = u, A_{a_N-1} = v - qN^2 + z - y) = \mathbf{P}_{\beta,0}(X_{a_N} = u - y, A_{a_N-1} = v - qN^2 + z - y\, a_N),$$

so the sum in (5.35) can be written as

$$\mathbf{E}_{\beta,0}\Big[\exp\Big(\psi_{N,h_N^q}\big(A_{a_N-1} + qN^2 - z + y\, a_N,\ X_{a_N} + y\big) - \psi_{N_1,h_{N_1}^q}(qN^2 - z, y)\Big)\Big]. \tag{5.36}$$

Furthermore, we claim that for all $q \in \mathbb{R}$,

$$\mathcal{L}\big(\widetilde{h}^q/2\big) - q\,\widetilde{h}^q - \mathcal{G}\big(\widetilde{h}^q\big) = 0. \tag{5.37}$$

Indeed,

$$\begin{aligned}
\mathcal{L}\big(\widetilde{h}^q/2\big) - \mathcal{G}\big(\widetilde{h}^q\big) &= -\int_0^1 \mathcal{L}\big(\widetilde{h}^q(\tfrac{1}{2} - y)\big)\,dy + \mathcal{L}\big(\widetilde{h}^q/2\big) \\
&= \int_0^1\int_y^1 \widetilde{h}^q \mathcal{L}'\big(\widetilde{h}^q(u - \tfrac{1}{2})\big)du\,dy \\
&= \widetilde{h}^q \int_0^1 u\mathcal{L}'\big(\widetilde{h}^q(u - \tfrac{1}{2})\big)du \\
&= \widetilde{h}^q \quad \mathcal{G}'\big(\widetilde{h}^q\big) = \widetilde{h}^q \quad q,
\end{aligned}$$

where we have used that $\mathcal{L}$ is even to obtain the second line and that $\widetilde{h}^q$ is solution in h of $\mathcal{G}'(h) = q$ (recall Step 1) to obtain the last line.

Recollecting (5.35–5.37), Proposition 5.1 and the definition of $\mathcal{G}$ (5.9), we deduce from a direct computation that there exists $N_0 \in \mathbb{N}$ and a $C_1 > 0$ uniform in $q \in [q_1, q_2] \cap \frac{\mathbb{N}}{N^2}$, $N \geq N_0$ and $|y| \leq N/a_N$, $|z| \leq N^2/a_N$ such that

$$\begin{aligned}
&\mathbf{P}_{N,h_N^q}(X_{N_1} = y, A_{N_1-1} = qN^2 - z) \\
&\quad\leq \frac{C_1}{N^2} e^{-a_N \mathcal{L}(\widetilde{h}^q/2)}\, \mathbf{E}_{\beta,0}\Big[\exp\Big(2\widetilde{h}^q \frac{A_{a_N-1}}{N} - \frac{\widetilde{h}^q}{2} X_{a_N}\big(1 - \tfrac{2}{N}\big)\Big)\Big].
\end{aligned} \tag{5.38}$$

Finally, we conclude the proof of Lemma 5.11 by showing that the latter factor is bounded from above by $e^{a_N \mathcal{L}(\widetilde{h}^q/2)}$. Let $(U_j)_{j\geq 1}$ be the increments of X, and recall $A_{a_N} = \sum_{i=1}^{a_N}(a_N + 1 - i)U_i$. Then,

$$\begin{aligned}\mathbf{E}_{\beta,0}\Big[\exp\Big(2\widetilde{h}^q \frac{A_{a_N-1}}{N} - \frac{\widetilde{h}^q}{2} X_{a_N}\big(1-\tfrac{2}{N}\big)\Big)\Big] &= \mathbf{E}_{\beta,0}\Bigg[\exp\Bigg(\sum_{j=1}^{a_N} U_j\,\widetilde{h}^q\big(-\tfrac{1}{2} + \tfrac{2}{N}(a_N + \tfrac{3}{2} - j)\big)\Bigg)\Bigg] \\ &= \prod_{j=1}^{a_N} \exp\big(\mathcal{L}(\widetilde{h}^q\big(-\tfrac{1}{2} + \tfrac{2}{N}(a_N + \tfrac{3}{2} - j)\big))\big) \\ &\leq e^{a_N \mathcal{L}(\widetilde{h}^q/2)}\,, \end{aligned} \tag{5.39}$$

where we used that $\mathcal{L}$ is even and increasing on $[0, \beta/2)$ (recall (2.2)). □

5.4 Step 4: Proof of Lemma 5.8

Outline of the proof of Lemma 5.8. Recall (5.16–5.17). For conciseness, from now on we set $h := \widetilde{h}^q/2$, $\bar{x} = (x_1, x_2)$ and $\bar{a} = (a_1, a_2)$, and we also define

$$\mathcal{H}_N \;:=\; \{(\bar{x}, \bar{a}) \in \mathcal{C}_N^2 \times \mathcal{D}_N^2\}\,. \tag{5.40}$$

Let us decompose $M_{N,q}$ by summing over possible values of $X_{a_N}, X_{N-a_N} \in \mathcal{C}_N$ and $A_{a_N}, A_N - A_{N-a_N} \in \mathcal{D}_N$. Using Markov property at times a_N and $N - a_N$, and applying time-reversibility on the part of the walk between times $N - a_N$ and N (i.e., (5.20) with $n = a_N$ and $h = 0$), we may write

$$M_{N,q} \;=\; \sum_{(\bar{x},\bar{a})\in\mathcal{H}_N} R_N(x_1, a_1)\, T_N(\bar{x}, \bar{a})\, R_N(x_2, a_2)\,, \tag{5.41}$$

with

$$R_N(x, a) \;:=\; \mathbf{P}_{\beta,0}(X_{[1,a_N]} > 0,\, X_{a_N} = x,\, A_{a_N} = a)\,, \tag{5.42}$$

and, after setting $N_2 = N - 2a_N$,

$$T_N(\bar{x}, \bar{a}) \;:=\; \mathbf{P}_{\beta,0}\big(X_{N_2} = x_2 - x_1,\; A_{N_2-1} = qN^2 - a_1 - a_2 - x_1(N_2 - 1),\; X_{[0,N_2]} > -x_1\big)\,. \tag{5.43}$$

Then, the proof of Lemma 5.8 is divided into three steps.

(a) We perform a change of measure in $R_N(x_1, a_1)$, $R_N(x_2, a_2)$ by tilting every increment $(X_{i+1} - X_i)_{i=0}^{a_N-1}$ uniformly with $\widetilde{\mathbf{P}}_h$, and in $T_N(\bar{x}, \bar{a})$ by applying the tilting introduced in (5.4) with coefficients $(N_2, h^q_{N_2})$. We prove that the combined exponential factors given by those changes of measure are equivalent to $\exp(-N\widetilde{\psi}(q))$.

(b) Then we estimate $T_N(\bar{x}, \bar{a})$ under its tilted distribution by using the local limit theorem for the vector (X_{N_2}, A_{N_2-1}) displayed in Proposition 6.1, and by observing that the additional constraint $X_{[0,N_2]} > -x_1$ does not alter the asymptotics.

(c) Finally, after recombining the sum in $M_{N,q}$, we observe that it only remains to estimate the probability that a random walk under the biased law $\widetilde{\mathbf{P}}_h$ remains positive, which has an explicit, positive limit.

Step (a). We begin with $R_N(x_1, a_1)$ and $R_N(x_2, a_2)$. Recalling (2.7), we perform a change of measure by tilting every increment $(X_{i+1} - X_i)_{i=0}^{a_N-1}$ with $\widetilde{\mathbf{P}}_h$, that is for $(x, a) \in$

$\{(x_1, a_1), (x_2, a_2)\}$,

$$R_N(x, a) := e^{-hx + a_N \mathcal{L}(h)} \widetilde{\mathbf{P}}_h(X_i > 0, \forall i \le a_N, X_{a_N} = x, A_{a_N} = a)\,. \tag{5.44}$$

For the second factor $T_N(\bar{x}, \bar{a})$ we apply the tilting introduced in (5.4) and we obtain

$$T_N(\bar{x}, \bar{a}) = G^q_{N,\bar{x},\bar{a}}\, e^{-h^q_{N_2}\left(\frac{qN^2 - a_1 - a_2}{N_2} - x_1\left(1 - \frac{1}{N_2}\right)\right)}\, e^{+\frac{h^q_{N_2}}{2}\left(1 - \frac{1}{N_2}\right)(x_2 - x_1) + N_2\, \mathcal{G}_{N_2}(h^q_{N_2})}\,, \tag{5.45}$$

with

$$G^q_{N,\bar{x},\bar{a}} := \mathbf{P}_{N_2, h^q_{N_2}}\left[\Lambda_{N_2} = \left(\frac{qN^2 - a_1 - a_2}{N_2} - x_1\left(1 - \frac{1}{N_2}\right),\ x_2 - x_1\right),\ X_{[0,N_2]} > -x_1\right]. \tag{5.46}$$

Gathering (5.41), (5.44) and (5.45), we obtain

$$\begin{aligned} M_{N,q} = \sum_{(\bar{x},\bar{a}) \in \mathcal{H}_N} & e^{B^1_{N,q}(\bar{a}) + B^2_{N,q}(\bar{x})}\, \widetilde{\mathbf{P}}_h(X_{[1,a_N]} > 0, X_{a_N} = x_1, A_{a_N} = a_1)\, G^q_{N,\bar{x},\bar{a}} \\ & \times \widetilde{\mathbf{P}}_h(X_{[1,a_N]} > 0, X_{a_N} = x_2, A_{a_N} = a_2)\,, \end{aligned} \tag{5.47}$$

with

$$\begin{aligned} B^1_{N,q}(\bar{a}) &:= N_2\, \mathcal{G}_{N_2}(h^q_{N_2}) + 2\, a_N\, \mathcal{L}(h) - h^q_{N_2}\, \frac{qN^2 - a_1 - a_2}{N_2}\,, \\ B^2_{N,q}(\bar{x}) &:= (x_1 + x_2)\left[\frac{h^q_{N_2}}{2}\left(1 - \frac{1}{N_2}\right) - h\right]. \end{aligned} \tag{5.48}$$

Remark 5.12 We claim that the expected values of X_{a_N} under $\widetilde{\mathbf{P}}_h$ and $\mathbf{P}_{N,h^q_N}$ are very close. Indeed, notice that $\widetilde{\mathbf{E}}_h(X_1) = \mathcal{L}'(h)$ (so $\widetilde{\mathbf{E}}_h(X_{a_N}) = a_N \mathcal{L}'(h)$), so recalling Remark 5.6 and (5.11), we write

$$\begin{aligned} \left|a_N \mathcal{L}'(h) - \mathbf{E}_{N,h^q_N}(X_{a_N})\right| &= \left|\sum_{i=1}^{a_N} \mathcal{L}'(h) - \widetilde{\mathbf{E}}_{h^q_N(1 - \frac{i}{N}) - \frac{h^q_N}{2}(1 - \frac{1}{N})}(X_1)\right| \\ &\le \sum_{i=1}^{a_N} \left|\mathcal{L}'\left(\frac{\widetilde{h}^q}{2}\right) - \mathcal{L}'\left(\frac{h^q_N}{2} + \frac{h^q_N}{2}\left(\frac{1 - 2i}{N}\right)\right)\right| \\ &\le \sum_{i=1}^{a_N} \max_{x \in \left[\frac{K}{2}, \frac{\beta - K}{2}\right]} \left|\mathcal{L}''(x)\right| \left(\left|\frac{\widetilde{h}^q - h^q_N}{2}\right| + \beta\, \frac{a_N}{N}\right) \\ &\le (\text{const.})\left(\frac{a_N}{N^2} + \frac{(a_N)^2}{N}\right), \end{aligned} \tag{5.49}$$

which goes to 0 as $N \to \infty$. In particular, there exist $c_1, c_2, N_0 > 0$ such that $c_1 \quad a_N \le x \le c_2 \quad a_N$ for every $x \in \mathcal{C}_N$, $q \in [q_1, q_2]$ and $N \ge N_0$ (recall (5.16), and notice that $\mathcal{L}'(h)$ is uniformly bounded away from 0). Moreover, a similar argument yields that $\mathbf{E}_{N,h^q_N}(A_{a_N})$ remains close to $\mathcal{L}'(h) a_N^2/2$, hence every $a \in \mathcal{D}_N$ satisfies $|a| \le (\text{const.})\, a_N^2$ (we do not write the details here).

Recalling (2.5), let us prove that

$$B^1_{N,q}(\bar{a}) + B^2_{N,q}(\bar{x}) \;=\; -N \quad \widetilde{\psi}(q) + o(1)\,, \tag{5.50}$$

Henceforth, we drop the $\bar{a}$ and $\bar{x}$ dependency of $B^1_{N,q}(\bar{a})$ and $B^2_{N,q}(\bar{x})$ for conciseness. Let us first consider $B^2_{N,q}$. We recall that $h = \widetilde{h}^q/2$, thus

$$B^2_{N,q} = \frac{x_1 + x_2}{2}\left(h^q_{N_2} - \widetilde{h}^q - \frac{h^q_{N_2}}{N_2}\right), \tag{5.51}$$

which allows us to write the upper bound

$$|B^2_{N,q}| \le \frac{|x_1| + |x_2|}{2}\left(\left|h^q_{N_2} - \widetilde{h}^q\right| + \left|\frac{h^q_{N_2}}{N_2}\right|\right) \le (\text{const.})\frac{a_N}{N_2}. \tag{5.52}$$

where we used Proposition 5.1, Remark 5.12 and that $|h^q_{N_2}| < \beta$. Therefore, $B^2_{N,q}$ converges to 0 as $N \to \infty$ uniformly in $q \in [q_1, q_2]$.

Now, we consider $B^1_{N,q}$. We recall the definition of $\mathcal{D}_N$ in (5.16), and using Remark 5.12 we write on the one hand

$$\frac{qN^2 - a_1 - a_2}{N_2} = qN + 2 \quad q \quad a_N + O\Big(\frac{(a_N)^2}{N}\Big), \tag{5.53}$$

and on the other hand,

$$\begin{aligned} &N_2\,\mathcal{G}_{N_2}(h^q_{N_2}) + 2\,a_N\,\mathcal{L}(\widetilde{h}^q/2) - h^q_{N_2}\,(q \quad N + 2 \quad q \quad a_N) \\ &= N_2\Big[\mathcal{G}(\widetilde{h}^q) - \widetilde{h}^q \quad q + O\big(\tfrac{1}{(N_2)^2}\big)\Big] + 2\,a_N\Big[\mathcal{L}(\widetilde{h}^q/2) - 2\,\widetilde{h}^q \quad q + O\big(\tfrac{1}{(N_2)^2}\big)\Big] \\ &= N\big[\mathcal{G}(\widetilde{h}^q) - \widetilde{h}^q \quad q\big] + 2a_N\big[\mathcal{L}(\widetilde{h}^q/2) - \widetilde{h}^q \quad q - \mathcal{G}(\widetilde{h}^q)\big] + O(\tfrac{1}{N_2}), \end{aligned} \tag{5.54}$$

which, by (5.10) and (5.11), holds uniformly in $N \ge N_0$ and $q \in [q_1, q_2] \cap \frac{\mathbb{N}}{N^2}$, provided N_0 is chosen large enough. Plugging together (5.53) and (5.54), and recalling (5.37), we obtain $B^1_{N,q} = -N\,\widetilde{\psi}(q) + o(1)$, which finishes the proof of (5.50).

Step (b). Recall (5.46): the aim of this step is to prove the following lemma.

Lemma 5.13 *For $\beta > 0$,*

$$G^q_{N,\bar{x},\bar{a}} = \frac{\big(\vartheta(\widetilde{h}^q)\big)^{-\frac{1}{2}}}{2\pi N^2}(1 + o(1)), \tag{5.55}$$

where $o(1)$ is a function that converges to 0 as $N \to \infty$ uniformly in $q \in [q_1, q_2] \cap \frac{\mathbb{N}}{N^2}$ and $(\bar{x}, \bar{a}) \in \mathcal{H}_N$.

Proof Let us first relax the constraint $\{X_{[0,N_2]} > -x_1\}$ in $G^q_{N,\bar{x},\bar{a}}$ to define

$$\bar{G}^q_{N,\bar{x},\bar{a}} := \mathbf{P}_{N_2, h^q_{N_2}}\Big[\Lambda_{N_2} = \Big(\tfrac{qN^2 - a_1 - a_2}{N_2} - x_1\big(1 - \tfrac{1}{N_2}\big),\, x_2 - x_1\Big)\Big]. \tag{5.56}$$

Using the local limit theorem from Proposition 6.1, we prove that $\bar{G}^q_{N,\bar{x},\bar{a}}$ satisfies (5.55). Indeed, defining $y = q(N^2 - (N_2)^2) - a_1 - a_2 - x_1\frac{N_2 - 1}{N_2}$ and $z = x_2 - x_1$, then Proposition 6.1 implies

$$\sup_{q \in [q_1, q_2] \cap \frac{\mathbb{N}}{N^2}} \left|(N_2)^2 \bar{G}^q_{N,\bar{x},\bar{a}} - f_{\widetilde{\boldsymbol{h}}(q,0)}\Big(\tfrac{y}{(N_2)^{3/2}}, \tfrac{z}{\sqrt{N_2}}\Big)\right| \underset{N \to \infty}{\longrightarrow} 0, \tag{5.57}$$

where for $\boldsymbol{h} \in \mathcal{D}_\beta$, $f_{\boldsymbol{h}}$ is the density function of a bivariate, centered Gaussian vector with covariance matrix Hess $\mathcal{L}_\Lambda(\boldsymbol{h})$ (see Appendix A.1 for more details). Moreover, notice that $N^2 - (N_2)^2 \le 4\,N\,a_N$. Thus, Remark 5.12 yields that, for some C, N_0 sufficiently large,

$$\frac{|y|}{(N_2)^{3/2}} \le C\frac{a_N}{\sqrt{N}} \quad \text{and} \quad \frac{|z|}{\sqrt{N_2}} \le C\frac{a_N}{\sqrt{N}}, \qquad \text{for } (\bar{x}, \bar{a}) \in \mathcal{H}_N,\ q \in [q_1, q_2],\ N \ge N_0. \tag{5.58}$$

Recalling $a_N = o(\sqrt{N})$ and Remark 6.2, we have the uniform convergence

$$\sup_{q\in[q_1,q_2]} \sup_{(\bar{x},\bar{a})\in\mathcal{H}_N} \left| f_{\widetilde{\boldsymbol{h}}(q,0)}\left(\tfrac{y}{(N_2)^{3/2}}, \tfrac{z}{\sqrt{N_2}}\right) - f_{\widetilde{\boldsymbol{h}}(q,0)}(0,0)\right| \underset{N\to\infty}{\longrightarrow} 0. \tag{5.59}$$

At this stage, we complete the proof of (5.55) for $\bar{G}^{q}_{N,\bar{x},\bar{a}}$ by observing that $N_2 \sim N$ as $N \to \infty$, that $f_{\widetilde{\boldsymbol{h}}(q,0)}(0,0) = \vartheta(\widetilde{h}^q)^{-1/2}\,\frac{1}{2\pi}$ and by combining (5.57) and (5.59) with the fact that $f_{\widetilde{\boldsymbol{h}}(q,0)}$ is bounded from above on $\mathbb{R}^2$, uniformly in $q \in [q_1, q_2]$.

Regarding $G^{q}_{N,\bar{x},\bar{a}}$, let us define

$$\widehat{G}^{q}_{N,\bar{x}} = \sum_{i=1}^{N_2-1} \mathbf{P}_{N_2,h^q_{N_2}}\left(X_{N_2} = x_2 - x_1,\, X_i \le -c_1 a_N\right), \tag{5.60}$$

where c_1 is defined in Remark 5.12; hence

$$\bar{G}^{q}_{N,\bar{x},\bar{a}} \ge G^{q}_{N,\bar{x},\bar{a}} \ge \bar{G}^{q}_{N,\bar{x},\bar{a}} - \widehat{G}^{q}_{N,\bar{x}}. \tag{5.61}$$

Thereby, the proof of Lemma 5.13 will be complete once we show that $N^2\,\widehat{G}^{q}_{N,\bar{x}} = o(1)$. Since $|x_1 - x_2| \le 2(a_N)^{3/4}$ we have that, for N large enough, $-c_1 a_N + x_1 - x_2 \le -c_1 a_N/2$. Thus, for $i \in \{\frac{N_2}{2}, \ldots, N_2 - 1\}$, the time-reversal property (5.20) (with $n = N_2$ and $h = h^q_{N_2}$) gives

$$\mathbf{P}_{N_2,h^q_{N_2}}\left(X_{N_2} = x_2 - x_1,\, X_i \le -c_1 a_N\right) \le \mathbf{P}_{N_2,h^q_{N_2}}\left(X_{N_2-i} \le -\tfrac{c_1}{2} a_N\right). \tag{5.62}$$

Coming back to (5.60) and using (5.62) for $i \ge N_2/2$ we can bound $\widehat{G}^{q}_{N,\bar{x}}$ from above as

$$\widehat{G}^{q}_{N,\bar{x}} \le 2\sum_{i=1}^{N_2/2} \mathbf{P}_{N_2,h^q_{N_2}}\left(X_i \le -\tfrac{c_1}{2} a_N\right) \le 2\, e^{-\frac{\lambda c_1 a_N}{2}} \sum_{i=1}^{N_2/2} \mathbf{E}_{N_2,h^q_{N_2}}\left(e^{-\lambda X_i}\right), \tag{5.63}$$

for some $\lambda > 0$, where we used a Chernov inequality. We claim that the latter sum is uniformly bounded, therefore this concludes the proof of Lemma 5.13 (recall that $a_N \gg \log N$). This follows directly from the following lemma, which was already proven in [4] (we do not reproduce the computations here). □

Lemma 5.14 [4,Lemma 6.2] *For $[q_1, q_2] \subset (0, \infty)$, there exist three positive constants C', C_1, λ such that for N large enough, the following bound holds true*

$$\mathbf{E}_{N,h^q_N}\left[e^{-\lambda X_j}\right] \le C'\, e^{-C_1 j}, \quad \textit{for} \quad j \le \tfrac{N}{2} \textit{ and } q \in [q_1, q_2] \cap \tfrac{\mathbb{N}}{N^2}.$$

Step (c). By plugging (5.50) and Lemma 5.13 into (5.47), we obtain

$$M_{N,q} = (1 + o(1))\, \frac{\left(\vartheta(\widetilde{h}^q)\right)^{-\frac{1}{2}}}{2\pi N^2}\, e^{-N\widetilde{\psi}(q)}\, \widehat{M}_{N,q}, \tag{5.64}$$

with

$$\begin{aligned}\widehat{M}_{N,q} &:= \sum_{(\bar{x},\bar{a})\in\mathcal{H}_N} \widetilde{\mathbf{P}}_h(X_{[1,a_N]} > 0,\, X_{a_N} = x_1,\, A_{a_N} = a_1)\, \widetilde{\mathbf{P}}_h(X_{[1,a_N]} > 0,\, X_{a_N} = x_2,\, A_{a_N} = a_2)\\ &= \left[\widetilde{\mathbf{P}}_h(X_{[1,a_N]} > 0,\, X_{a_N} \in \mathcal{C}_N,\, A_{a_N} \in \mathcal{D}_N)\right]^2, \end{aligned} \tag{5.65}$$

where we recall the definitions of $\mathcal{C}_N, \mathcal{D}_N$ from (5.16), and that $h = \widetilde{h}^q/2$. Recall also that we defined $\kappa(x) = \widetilde{\mathbf{P}}_x(X_{[1,\infty]} > 0)$ for $x \in [0, \beta/2)$ (see (2.8)). Therefore, the proof of Lemma 5.8 is completed by the following lemma, and by recalling from Remark 5.6 that h is bounded away from 0 uniformly in $q \in [q_1, q_2]$. Lemma 5.15 is proven afterwards. □

Lemma 5.15 (i) *For $\beta > 0$ and $x \in [0, \beta/2)$, one has*

$$\kappa(x) = \frac{e^{2x} - 1}{e^{x+\beta/2} - 1}, \tag{5.66}$$

in particular κ is continuous, increasing on $[0, \beta/2)$, and positive on $(0, \beta/2)$.

(ii) *For $\beta > 0$, one has*

$$\widetilde{\mathbf{P}}_h(X_{[1,a_N]} > 0, X_{a_N} \in \mathcal{C}_N, A_{a_N} \in \mathcal{D}_N) = \kappa(h) + o(1), \tag{5.67}$$

where $o(1)$ is a function that converges to 0 as $N \to \infty$ uniformly in $q \in [q_1, q_2]$.

Proof of Lemma 5.15 (i) Pick $x \in [0, \beta/2)$ and define $\rho = \inf\{i \geq 1, X_i \leq 0\}$, hence

$$1 - \kappa(x) = \widetilde{\mathbf{P}}_x(\rho < \infty) = \mathbf{E}_\beta\big[e^{xX_\rho - \mathcal{L}(x)\rho} 1_{\{\rho<\infty\}}\big] = \mathbf{E}_\beta\big[e^{xX_\rho - \mathcal{L}(x)\rho}\big],$$

where we used (2.7) and that $\rho < \infty$ $\mathbf{P}_\beta$-a.s.. As claimed in the proof of Lemma 3.2, ρ and X_ρ are independent, and $-X_\rho$ follows a geometric law on $\mathbb{N} \cup \{0\}$ with parameter $1 - e^{-\beta/2}$. Similarly to (3.24), $(e^{-xX_n - \mathcal{L}(x)n})_{n\geq 1}$ is a martingale under $\mathbf{P}_\beta$ (recall that $\mathcal{L}$ is even), and it is uniformly integrable when stopped at time ρ; thus Doob's Optional Stopping Theorem yields that

$$\mathbf{E}_\beta\big[e^{-\mathcal{L}(x)\rho}\big] = \mathbf{E}_\beta\big[e^{-xX_\rho}\big]^{-1}.$$

Thereby,

$$1 - \kappa(x) = \widetilde{\mathbf{P}}_x(\rho < \infty) = \mathbf{E}_\beta\big[e^{xX_\rho}\big]\mathbf{E}_\beta\big[e^{-xX_\rho}\big]^{-1} = \frac{1 - e^{x-\beta/2}}{1 - e^{-x-\beta/2}},$$

which yields (5.66).

(ii) We write

$$\begin{aligned}&\Big|\widetilde{\mathbf{P}}_h(X_{[1,a_N]} > 0, X_{a_N} \in \mathcal{C}_N, A_{a_N} \in \mathcal{D}_N) - \kappa(h)\Big|\\ &\quad\leq \Big|\widetilde{\mathbf{P}}_h(X_{[1,a_N]} > 0) - \kappa(h)\Big| + \widetilde{\mathbf{P}}_h(X_{a_N} \notin \mathcal{C}_N) + \widetilde{\mathbf{P}}_h(A_{a_N} \notin \mathcal{D}_N),\end{aligned} \tag{5.68}$$

and we claim that all three terms in the r.h.s. above go to 0 as $N \to \infty$ uniformly in $q \in [q_1, q_2]$.

We start with the first term in (5.68). Recalling Remark 5.6, there exist $K > 0$ such that $h \in [K, \beta - K]$ for all $q \in [q_1, q_2]$. Then we write with a Chernov inequality,

$$0 \leq \widetilde{\mathbf{P}}_h(X_{[1,a_N]} > 0) - \kappa(h) \leq \sum_{j=a_N+1}^{\infty} \widetilde{\mathbf{P}}_h(X_j \leq 0) \leq \sum_{j=a_N+1}^{\infty} e^{-(\mathcal{L}(h)-\mathcal{L}(h-K/2))j}, \tag{5.69}$$

and since $\mathcal{L}$ is convex and increasing, $\mathcal{L}(h) - \mathcal{L}(h - K/2) \geq \mathcal{L}'(K/2)\frac{K}{2} > 0$. Therefore, the r.h.s. in (5.69) converges to 0 as $N \to \infty$ uniformly in $q \in [q_1, q_2]$.

Regarding the second term in (5.68), (5.16) and Remark 5.12 imply for N sufficiently large,

$$\{X_{a_N} \notin \mathcal{C}_N\} \subset \{|X_{a_N} - \mathcal{L}'(h) \ \ a_N| \geq \tfrac{1}{2}(a_N)^{3/4}\}, \tag{5.70}$$

where we recall $\widetilde{\mathbf{E}}_h(X_1) = \mathcal{L}'(h)$. Then, Chernov inequality gives for $t > 0$,

$$\widetilde{\mathbf{P}}_h(X_{a_N} \geq \mathcal{L}'(h) \ \ a_N + \tfrac{1}{2}(a_N)^{3/4}) \leq \exp\Big(a_N\big(\mathcal{L}(h+t) - \mathcal{L}(h) - t\mathcal{L}'(h)\big) - t\tfrac{1}{2}a_N^{3/4}\Big). \tag{5.71}$$

Recalling (2.2) and Remark 5.6, there exists $c > 0$ such that $\mathcal{L}(h+t)-\mathcal{L}(h)-t\mathcal{L}'(h) \leq c \quad t^2$ for t small enough, uniformly in $q \in [q_1, q_2]$. Letting $\varepsilon > 0$ and $t = a_N^{-1/4-\varepsilon}$, we obtain for N sufficiently large and some $c' > 0$,

$$\widetilde{\mathbf{P}}_h(X_{a_N} \geq \mathcal{L}'(h) \quad a_N + \tfrac{1}{2}(a_N)^{3/4}) \leq e^{-c' a_N^{1/2-\varepsilon}}, \tag{5.72}$$

uniformly in $q \in [q_1, q_2]$. The same bound holds for the event $\{X_{a_N} \leq \mathcal{L}'(h)\, a_N - \frac{1}{2}(a_N)^{3/4}\}$ (we do not write the details again), and recollecting (5.70) this proves that the second term in (5.68) goes to 0 as $N \to \infty$ uniformly in $q \in [q_1, q_2]$.

Finally, the third term in (5.68) is handled very similarly, first by observing that (5.16) and Remark 5.12 imply

$$\{A_{a_N} \notin \mathcal{D}_N\} \subset \{|A_{a_N} - \mathcal{L}'(h) \quad a_N^2/2| \geq \tfrac{1}{2}(a_N)^{7/4}\}, \tag{5.73}$$

then by writing a Chernov inequality with $A_{a_N} = \sum_{i=1}^{a_N}(a_N+1-i) \quad U_i$, where $(U_i)_{i=1}^{a_N} \sim \widetilde{\mathbf{P}}_h$ are i.i.d.. Details are left to the reader. □

6 Proof of Theorem 2.2

We recall (3.8–3.11) and Remark 3.1. For an event A we denote by $Z_{L,\beta}(A)$ the partition function restricted to trajectories in $A \cap \Omega_L$, i.e.,

$$Z_{L,\beta}(A) = \sum_{\ell \in A \cap \Omega_L} e^{H_{L,\beta}(\ell)}.$$

For $L \in \mathbb{N}$, the function $k \mapsto P_{L,\beta}\big(|I_{\max}(\ell)| \geq L-k\big)$ is non decreasing. Therefore, proving (2.15) with $P_{L,\beta}\big(|I_{\max}(\ell)| \geq L-3k\big)$ implies the result. Pick $\ell \in \Omega_L$, and let $\mathcal{J}_k := \max\{j \geq 0 \colon \mathfrak{X}_j \leq k\}$. Note that, if $\ell \in \Omega_L$ has no bead starting in $\{k, \ldots, L-k\}$ and if the last bead of ℓ starting before k begins with at most $k-1$ horizontal steps, then the longest sequence of non-zero vertical stretches of alternating signs in ℓ has a total length at least $L-3k$, i.e.,

$$A_{L,k} \cap B_{L,k} \subset \{|I_{\max}(\ell)| \geq L-3k\}. \tag{6.1}$$

with

$$\begin{aligned} A_{L,k} &:= \{\ell \in \Omega_L \colon \mathfrak{X} \cap \{k, \ldots, L-k\} = \emptyset\} \\ B_{L,k} &:= \{\ell \in \Omega_L \colon \exists j \in \{1, \ldots, k\} \colon \ell_{\tau_{\mathcal{J}_k}+j} \neq 0\}. \end{aligned} \tag{6.2}$$

Since $k \mapsto P_{L,\beta}\big(|I_{\max}(\ell)| \geq L-3k\big)$ is non decreasing and since (6.1) yields

$$\{|I_{\max}(\ell)| \geq L-3k\}^c \subset (A_{L,k})^c \cup \big(A_{L,k} \cap (B_{L,k})^c\big),$$

it follows that Theorem 2.2 will be proven once we show that for every $\varepsilon > 0$ there exists a $k_\varepsilon \in \mathbb{N}$ and a $L_\varepsilon \in \mathbb{N}$ such that on the one hand $L_\varepsilon \geq 3k_\varepsilon$ and on the other hand, for $L \geq L_\varepsilon$,

$$P_{L,\beta}\big((A_{L,k_\varepsilon})^c\big) \leq \varepsilon \quad \text{and} \quad P_{L,\beta}\big(A_{L,k_\varepsilon} \cap (B_{L,k_\varepsilon})^c\big) \leq \varepsilon. \tag{6.3}$$

For simplicity let us write $\alpha = -\widetilde{\mathcal{G}}(a_\beta)$. A straightforward consequence of Theorem 2.1 and Corollary 4.2 is that there exists $0 < C_1 < C_2 < \infty$ such that for every $L \in \mathbb{N}$,

$$\frac{C_1}{L^{3/4}}\, e^{\beta L - \alpha\sqrt{L}} \leq \widehat{Z}^{\circ}_{L,\beta} \leq Z_{L,\beta} \leq \frac{C_2}{L^{3/4}}\, e^{\beta L - \alpha\sqrt{L}}. \tag{6.4}$$

Thus for $k, L \in \mathbb{N}$ such that $L \geq 3k$ we can write

$$\begin{aligned}
P_{L,\beta}\big((A_{L,k})^c\big) &\leq \sum_{j=k}^{L-k} P_{L,\beta}\big(j \in \mathfrak{X}\big) = \frac{1}{Z_{L,\beta}} \sum_{j=k}^{L-k} Z_{j,\beta}^{c}\, Z_{L-j,\beta}(\ell_1 \geq 0) \\
&\leq \frac{1}{Z_{L,\beta}} \sum_{j=k}^{L-k} Z_{j,\beta}\, Z_{L-j,\beta} \\
&\leq (\text{const.})\, e^{\alpha\sqrt{L}}\, L^{3/4} \sum_{j=k}^{L-k} \frac{1}{j^{3/4}}\, e^{-\alpha\sqrt{j}}\, \frac{1}{(L-j)^{3/4}}\, e^{-\alpha\sqrt{L-j}} \\
&\leq (\text{const.}) \sum_{j=k}^{L-k} \frac{L^{3/4}}{j^{3/4}\,(L-j)^{3/4}}\, e^{-\alpha(\sqrt{j}+\sqrt{L-j}-\sqrt{L})} \\
&\leq (\text{const.}) \sum_{j=k}^{L/2} \frac{1}{j^{3/4}}\, e^{-\frac{\alpha\sqrt{j}}{2}} \leq (\text{const.}) \sum_{j=k}^{\infty} \frac{1}{j^{3/4}}\, e^{-\frac{\alpha\sqrt{j}}{2}}, \qquad (6.5)
\end{aligned}$$

where in the second line we have used (6.4) and in the last line we have used the convex inequality: $\sqrt{L}-\sqrt{L-j} \leq \frac{1}{2}\sqrt{j}$ for $0 \leq j \leq L/2$. Since the r.h.s. in (6.5) does not depend on L and vanishes as $k \to \infty$, the leftmost inequality in (6.3) is proven.

Let us now deal with the second inequality in (6.3). For $L \geq 3k$, we partition the set $A_{L,k} \cap (B_{L,k})^c$ by recording r (respectively s), the rightmost (resp. leftmost) point in $\mathfrak{X}$ that is smaller than k (resp. larger than $L-k$). Moreover, the fact that $\ell \in (B_{L,k})^c$ implies that the bead which covers the interval $\{k, \dots, L-k\}$ begins with k zero-length vertical stretches. Thus,

$$\begin{aligned}
Z_{L,\beta}(A_{L,k} \cap (B_{L,k})^c) &= \sum_{r=0}^{k-1} \sum_{s=L-k+1}^{L} Z_{L,\beta}(\{\mathfrak{X} \cap \{r, \dots, s\} = \{r, s\}\} \cap (B_{L,k})^c) \\
&= \sum_{r=0}^{k-1} \sum_{s=L-k+1}^{L} Z_{r,\beta}^{c}\, \widehat{Z}_{s-r,\beta}^{\circ}(\ell_1 = \cdots = \ell_k = 0)\, Z_{L-s,\beta}(\ell_1 \geq 0) \\
&\leq 2 \sum_{r=0}^{k-1} \sum_{s=L-k+1}^{L} Z_{r,\beta}^{c}\, \widehat{Z}_{s-r-k,\beta}^{\circ}\, Z_{L-s,\beta}(\ell_1 \geq 0) \\
&= 2\, Z_{L-k,\beta}(\{\mathfrak{X} \cap \{r, \dots, s-k\} = \{r, s-k\}\}), \qquad (6.6)
\end{aligned}$$

where the third line in (6.6) is obtained by observing that $\widehat{Z}_{j,\beta}^{\circ}(\ell_1 = \cdots = \ell_k = 0) \leq 2\widehat{Z}_{j-k,\beta}^{\circ}$ for $j \geq k+2$. The factor 2 in the r.h.s. of the latter inequality comes from the fact that there is no constraint on the sign of the $(k+1)$-th stretch of $\ell \in \widehat{\Omega}_j^{\circ}$ satisfying $\ell_1 = \cdots = \ell_k = 0$. The r.h.s. in the fourth line of (6.6) is obviously bounded above by $2\, Z_{L-k,\beta}$ and therefore,

$$P_{L,\beta}\big(A_{L,k} \cap (B_{L,k})^c\big) = \frac{Z_{L,\beta}(A_{L,k} \cap (B_{L,k})^c)}{Z_{L,\beta}} \leq \frac{2\, Z_{L-k,\beta}}{Z_{L,\beta}}. \qquad (6.7)$$

It remains to use (6.4) so that (6.7) becomes

$$P_{L,\beta}\big(A_{L,k} \cap (B_{L,k})^c\big) \leq (\text{const.}) \quad \frac{L^{3/4}}{(L-k)^{3/4}} e^{-\beta k} e^{-\alpha(\sqrt{L-k}-\sqrt{L})}$$

$$\leq (\text{const.}) \quad e^{-\beta k} e^{-\alpha(\sqrt{L-k}-\sqrt{L})} \,. \tag{6.8}$$

For $\varepsilon > 0$ we pick $k_\varepsilon \in \mathbb{N}$ such that (const.) $e^{-\beta k_\varepsilon} \leq \varepsilon/2$. Moreover, a straightforward computation gives us that $\lim_{L\to\infty} \sqrt{L-k_\varepsilon} - \sqrt{L} = 0$. This completes the proof of the rightmost inequality in (6.3) and ends the proof of Theorem 2.2.

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A Appendix

A.1 Local Limit Theorem for Large Swiped Areas

In this section we provide a local limit theorem for random walks under distributions $\mathbf{P}_{n,h}$ (recall (5.4)), for which events such as $\{\frac{1}{n}\Lambda_n(X) = (q,0)\}$, $q > 0$ become typical. This theorem was proven in [8,Theorem 4.2] and then refined in [4,Proposition 6.1] by showing that the convergence holds uniformly in $q \in [q_1, q_2]$. It is required to prove Proposition 4.6, more precisely for Lemmata 5.7–5.8.

Recall (5.5–5.8), and set

$$\mathbf{B}(\boldsymbol{h}) \;:=\; \text{Hess}\, \mathcal{L}_\Lambda(\boldsymbol{h})\,, \quad \boldsymbol{h} \in \mathcal{D}_\beta\,. \tag{A.1}$$

For $q \in \mathbb{R}$, we consider the density function of a centered Gaussian vector distribution with covariance matrix $\mathbf{B}(\widetilde{\boldsymbol{h}}(q,0))$,

$$f_{\widetilde{\boldsymbol{h}}(q,0)}(\bar{x}) \;:=\; \frac{(\text{Det}(\mathbf{B}(\widetilde{\boldsymbol{h}}(q,0))))^{-1/2}}{2\pi} \exp\big(-\tfrac{1}{2}\langle \mathbf{B}(\widetilde{\boldsymbol{h}}(q,0))^{-1}\bar{x}\,,\bar{x}\rangle\big), \quad \bar{x} \in \mathbb{R}^2\,. \tag{A.2}$$

This latter expression is well-posed: indeed, from the definition of $\mathcal{L}$ (2.2) we deduce easily that $\mathcal{L}''(h) > 0$ for $h \in (-\beta/2, \beta/2)$. As a consequence, we can rule out the equality case when applying the Cauchy–Schwartz inequality to (2.6), so we deduce

$$\vartheta(\widetilde{h}^q) \;=\; \text{Det}(\mathbf{B}(\widetilde{\boldsymbol{h}}(q,0))) \;>\; 0\,, \quad q \in \mathbb{R}\,. \tag{A.3}$$

Proposition 6.1 [4,Proposition 6.1] *For* $[q_1, q_2] \subset (0,\infty)$*, we have*

$$\sup_{q\in[q_1,q_2]\cap\frac{\mathbb{N}}{n^2}} \; \sup_{y,z\in\mathbb{Z}} \big|n^2\, \mathbf{P}_{n,\,h_n^q}\big(A_{n-1} = n^2 q + y,\, X_n = z\big) - f_{\widetilde{\boldsymbol{h}}(q,0)}\big(\tfrac{y}{n^{3/2}},\, \tfrac{z}{\sqrt{n}}\big)\big| \underset{n\to\infty}{\longrightarrow} 0\,.$$

Remark 6.2 Recall (A.1–A.3). Since $\mathcal{L}_\Lambda$ is convex on $\mathcal{D}_\beta$, (A.3) is sufficient to assert that $\mathbf{B}(\widetilde{\boldsymbol{h}}(q,0))$ is a symmetric positive-definite matrix for every $q \in \mathbb{R}$. Thus, the eigenvalues of $\mathbf{B}(\widetilde{\boldsymbol{h}}(q,0))$ are positive and continuous in $q \in \mathbb{R}$. This yields that, for $[q_1, q_2] \subset (0,\infty)$, the eigenvalues of $\mathbf{B}(\widetilde{\boldsymbol{h}}(q,0))$ are bounded above and below by positive constants that are uniform in $q \in [q_1, q_2]$. Therefore, there exists a compact subset $\mathcal{K} \subset (0,\infty)$ such that $\vartheta(\widetilde{h}^q) \in \mathcal{K}$ for every $q \in [q_1, q_2]$. From (A.2), the latter implies that $f_{\widetilde{\boldsymbol{h}}(q,0)}$ is bounded from above on $\mathbb{R}^2$, uniformly in $q \in [q_1, q_2]$.

A.2 Proof of Proposition 5.4

Let us start with the case $j = 0$. We recall (5.3) and we set

$$f_{N,h}(x) := \mathcal{L}(\tfrac{h}{2}(1+\tfrac{1}{N}) - h\,\tfrac{x}{N})\,, \tag{A.4}$$

and therefore $N\mathcal{G}_N(h) = \sum_{i=1}^N f_{N,h}(i)$. We apply the Euler–Maclaurin summation formula (see e.g. [17,Theorem 0.7]) and since $f_{N,h}$ is $\mathcal{C}^2$ we obtain that

$$N\mathcal{G}_n(h) = A(N,h) + B(N,h)\,, \tag{A.5}$$

with

$$A(N,h) := \frac{f_{N,h}(1) + f_{N,h}(N)}{2} + \int_1^N f_{N,h}(x)\,dx\,, \tag{A.6}$$

and

$$B(N,h) := \frac{1}{2}\int_1^N f''_{N,h}(x)\,\big(B_2(0) - B_2(x - \lfloor x \rfloor)\big)\,dx\,, \tag{A.7}$$

where B_2 is the second Bernoulli polynomial.

We start by considering $A(N,h)$. Recalling the definition of $\mathcal{L}$ in (2.2) and the fact that $\mathbf{P}_\beta$ is symmetric, we claim that $\mathcal{L}$ is even and therefore

$$\frac{f_{N,h}(1) + f_{N,h}(N)}{2} = \mathcal{L}\Big(\frac{h}{2} - \frac{h}{2N}\Big)\,. \tag{A.8}$$

We recall the definition of $\mathcal{G}$ in (5.9) and a straightforward computation gives

$$\begin{aligned}\int_1^N f_{N,h}(x)\,dx &= N\int_{1/2N}^{1-1/2N} \mathcal{L}(h(\tfrac{1}{2} - y))\,dy\\ &= N\mathcal{G}(h) - 2N\int_0^{1/2N} \mathcal{L}(h(\tfrac{1}{2} - z))\,dz\,,\end{aligned} \tag{A.9}$$

where we have used the change of variable $y = x/N$ to get the first equality and the parity of $\mathcal{L}$ combined with the change of variable $z = 1 - y$ to obtain the second equality. Obviously, for N large enough and for every $h \in [-\beta + K, \beta - K]$ we have that both $\frac{h}{2}$ and $\frac{h}{2}(1 - \frac{1}{N})$ belong to $\mathcal{R}_K := \big[-\frac{\beta}{2} + \frac{K}{2}, \frac{\beta}{2} - \frac{K}{2}\big]$. Recalling Remark 5.6, we set $C'_K := \max\{|\mathcal{L}'(x)|, x \in \mathcal{R}_K\}$ and for N large enough

$$\Big|\mathcal{L}\Big(\frac{h}{2} - \frac{h}{2N}\Big) - \mathcal{L}\Big(\frac{h}{2}\Big)\Big| \le C'_K\,\frac{|h|}{2N} \le \beta\, C'_K\,\frac{1}{2N}\,, \tag{A.10}$$

and

$$\Big|2N\int_0^{1/2N} \mathcal{L}(h(\tfrac{1}{2} - z))dz - \mathcal{L}(\tfrac{h}{2})\Big| \le 2N\int_0^{1/2N} C'_K\,|h|\,z dz \le \beta\, C'_K\,\frac{1}{4N}\,. \tag{A.11}$$

Combining (A.8–A.11) we claim that, for N large enough

$$|A(N,h) - N\,\mathcal{G}(h)| \le \beta\, C'_K\,\tfrac{1}{N}\,, \quad \forall h \in [-\beta + K, \beta - K]\,. \tag{A.12}$$

It remains to consider $B(N,h)$ (recall (A.7)). To that aim, we compute the second derivative of $f''_{N,h}$ and obtain

$$f''_{N,h}(x) = \tfrac{h^2}{N^2}\,\mathcal{L}''\big[\tfrac{h}{2}(1 + \tfrac{1}{N}) - h\,\tfrac{x}{N}\big]\,, \quad x \in [1, N]\,. \tag{A.13}$$

It turns out that for $x \in [1, N]$ we have $|\frac{h}{2}(1+\frac{1}{N}) - h\frac{x}{N}| \le \frac{|h|}{2}$. Thus, $h \in [-\beta+K, \beta-K]$ yields $\frac{h}{2}(1+\frac{1}{N}) - h\frac{x}{N} \in \mathcal{R}_K$, so recalling Remark 5.6 we set $C''_K := \max\{|\mathcal{L}''(x)|, x \in \mathcal{R}_K\}$ and we obtain $|f''_{N,h}(x)| \le C''_K \beta^2/N^2$. As a consequence, we can use (A.7) to write

$$\big|B(N,h)\big| \le C''_K \frac{\beta^2}{N} \max\{|B_2(u)|, u \in [0,1]\}\,. \tag{A.14}$$

At this stage, it remains to combine (A.5), (A.12) and (A.14) to complete the proof of Proposition 5.4 in the case $j = 0$.

Regarding the case $j = 1$, the proof is very similar. We can repeat (A.4–A.7) with $N\mathcal{G}'_N(h)$ instead of $N\mathcal{G}_N(h)$ and after redefining $f_{N,h}$ as

$$f_{N,h}(x) := \big(\tfrac{1}{2}(1+\tfrac{1}{N}) - \tfrac{x}{N}\big)\,\mathcal{L}'[\tfrac{h}{2}(1+\tfrac{1}{N}) - h\,\tfrac{x}{N}]\,. \tag{A.15}$$

Using that $\mathcal{L}'$ is odd, introducing the function $g_h(u) := (\frac{1}{2}-u)\mathcal{L}'[h(\frac{1}{2}-u)]$ and computing $\mathcal{G}'$ from (5.9), we obtain that in this case

$$\begin{aligned} A(N,h) &= N\int_{1/2N}^{1-1/2N} (\tfrac{1}{2}-y)\,\mathcal{L}'[h(\tfrac{1}{2}-y)]\,dy + \tfrac{1}{2}\,(1-\tfrac{1}{N})\,\mathcal{L}'[h(\tfrac{1}{2}-\tfrac{1}{2N})] \\ &= N\mathcal{G}'(h) + g(\tfrac{1}{2N}) - 2N\int_0^{1/2N} g(u)\,du\,. \end{aligned} \tag{A.16}$$

Taking the derivative of g we obtain $g'(u) = -\mathcal{L}'[h(\frac{1}{2}-u)] - h(\frac{1}{2}-u)\mathcal{L}''[h(\frac{1}{2}-u)]$ so that

$$\sup_{u\in[0,1/2N]}\ \sup_{h\in[-\beta+K,\beta-K]} |g'(u)| \;\le\; C'_K + \beta\, C''_K\,. \tag{A.17}$$

Therefore,

$$\begin{aligned} \Big|g(\tfrac{1}{2N}) - 2N\int_0^{1/2N} g(u)\,du\Big| \;&\le\; \big|g(\tfrac{1}{2N}) - g(0)\big| + 2N\int_0^{1/2N} |g(u)-g(0)|\,du \\ &\le\; (C'_K + \beta C''_K)\frac{1}{N}\,. \end{aligned} \tag{A.18}$$

It remains to consider $B(N,h)$ for which we need to compute $f''_{N,h}$, i.e., for $x \in [1, N]$,

$$f''_{N,h}(x) = \tfrac{2h}{N^2}\,\mathcal{L}''\big[\tfrac{h}{2}(1+\tfrac{1}{N}) - h\tfrac{x}{N}\big] + \tfrac{h^2}{N^2}\,\big(\tfrac{1}{2}(1+\tfrac{1}{N}) - \tfrac{x}{N}\big)\mathcal{L}'''\big[\tfrac{h}{2}(1+\tfrac{1}{N}) - h\tfrac{x}{N}\big]. \tag{A.19}$$

Thus, after defining $C'''_K := \max\{|\mathcal{L}'''(x)|, x \in \mathcal{R}_K\}$ and by mimicking the former proof we obtain $|f'''_{N,h}(x)| \le (2\,C''_K\,\beta + \beta^2\,C'''_K)/N^2$ for $x \in [1, N]$. This is sufficient to claim (from (A.7)) that

$$\big|B(N,h)\big| \le (2\,C''_K\,\beta + \beta^2\,C'''_K)\,\frac{1}{N}\,\max\{|B_2(u)|, u \in [0,1]\}\,. \tag{A.20}$$

We combine (A.16), (A.18) and (A.20) and it completes the proof of Proposition 5.4 in the case $j = 1$. □

References

1. Binder, P.M., Owczarek, A.L., Veal, A.R., Yeomans, J.M.: Collapse transition in a simple polymer model: exact results. J. Phys. A **23**(18), L975–L979 (1990)
2. Brak, R., Guttmann, A.J., Whittington, S.G.: A collapse transition in a directed walk model. J. Phys. A **25**(9), 2437–2446 (1992)

3. Caravenna, F., Chaumont, L.: An invariance principle for random walk bridges conditioned to stay positive. Electron. J. Probab. **18**, 1–32 (2013)
4. Carmona, P., Nguyen, G.B., Pétrélis, N.: Interacting partially directed self avoiding walk. From phase transition to the geometry of the collapsed phase. Ann. Probab. **44**(5), 3234–3290 (2016)
5. Carmona, P., Nguyen, G.B., Pétrélis, N., Torri, N.: Interacting partially directed self-avoiding walk: a probabilistic perspective. J. Phys. A **51**(15), 153001 (2018)
6. Carmona, P., Pétrélis, N.: Interacting partially directed self avoiding walk: scaling limits. Electron. J. Probab. **21**(49), 1–52 (2016)
7. den Hollander, F.: Random Polymers, volume 1974 of Lecture Notes in Mathematics. Springer, Berlin, 2009. Lectures from the 37th Probability Summer School held in Saint-Flour (2007)
8. Dobrushin, R., Hryniv, O.: Fluctuations of shapes of large areas under paths of random walks. Probab. Theory Relat. Fields **105**(4), 423–458 (1996)
9. Foss, S., Korshunov, D., Zachary, S.: An Introduction to Heavy-Tailed and Subexponential Distributions, 2nd edn. Springer, New York (2013)
10. Guttmann, A.J.: Analysis of series expansions for non-algebraic singularities. J. Phys. A **48**(4), 045209 (2015)
11. Janse van Rensburg, E.J.: The Statistical Mechanics of Interacting Walks, Polygons, Animals and Vesicles. Oxford University Press, Oxford (2015)
12. Miracle-Sole, S., Ruiz, J.: On the Wulff Construction as a Problem of Equivalence of Statistical Ensembles, pp. 295–302. Springer, Boston (1994)
13. Nguyen, G.B., Petrelis, N.: A variational formula for the free energy of the partially directed polymer collapse. J. Stat. Phys. **151** (2012)
14. Owczarek, A.L., Prellberg, T., Brak, R.: New scaling form for the collapsed polymer phase. Phys. Rev. Lett. **70**, 951–953 (1993)
15. Perfilev, E., Wachtel, V.: Local tail asymptotics for the joint distribution of length and of maximum of a random walk excursion (2019). arXiv:1907.02578
16. Rockafellar, R.T.: Convex Analysis. Princeton University Press, Princeton (2015)
17. Tenenbaum, G.: Introduction to Analytic and Probabilistic Number Theory, 3rd ed. Amer. Math. Soc. (2015)
18. Zwanzig, R., Lauritzen, J.I.: Exact calculation of the partition function for a model of two dimensional polymer crystallization by chain folding. J. Chem. Phys. **48**(8), 3351 (1968)