

SURFACE TRANSITION IN THE COLLAPSED PHASE OF A SELF-INTERACTING WALK ADSORBED ALONG A HARD WALL

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The present paper is dedicated to the 2-dimensional Interacting Partially Directed Self Avoiding Walk constrained to remain in the upper-half plane and interacting with the horizontal axis. This model has originally been introduced to investigate the behavior of a homopolymer dipped in a poor solvent and adsorbed along a horizontal hard wall. It is known to undergo a *collapse* transition between an *extended* phase, inside which typical configurations of the polymer have a large horizontal extension (comparable to their total size), and a *collapsed* phase inside which the polymer looks like a globule.

It is conjectured in the physics literature (see, e.g., (*Phys. A, Stat. Mech. Appl.* **318** (2002) 171) or (*Phys. Rev. E* **65** (2002) 056124)) that inside the collapsed phase, a *surface* transition occurs between an *adsorbed-collapsed* regime where the bottommost layer of the globule is pinned at the hard wall, and a *desorbed-collapsed* regime where the globule wanders away from the wall. In the present paper, we consider a simplified “single-bead” version of the model, for which we establish rigorously the existence of the surface transition and exhibit its associated critical curve. To that aim, we display some sharp asymptotics of the partition function of this simplified model within the collapsed phase.

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Notation. Let \mathbb{N} be the set of positive integers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $(a_L)_{L \leq 1}$ and $(b_L)_{L \leq 1}$ be two sequences of positive numbers. We write that

$$(0.1) \quad a_L \sim_{L \rightarrow \infty} b_L \quad \text{if} \quad \lim_{L \rightarrow \infty} a_L/b_L = 1,$$

and also that

$$(0.2) \quad a_L \asymp b_L \quad \text{if} \quad c_1 b_L \leq a_L \leq c_2 b_L \quad \forall L \geq 1,$$

with c_1, c_2 two positive constants.

1. Introduction. In the present paper, we investigate a model for a $1 + 1$ dimensional polymer dipped in a poor solvent and simultaneously adsorbed along a horizontal hard wall. Although the model has attracted a continuous attention in the physics literature starting in the 1990s (see, e.g., [9, 10, 18]) until more recently (see, e.g., [21, 26] or [24]), it had, up to our knowledge, not been considered so far in the mathematical literature. This model interpolates between two families of polymer models that have been entirely solved in the last 20 years, that is, the wetting of a $1 + 1$ -dimensional random walk adsorbed along a hard wall (see, e.g., [6, 11] and [12]) and the collapse transition of the 2-dimensional Interacting Partially-Directed Self-avoiding Walk (IPDSAW) (see [5] for a review).

The coupling parameters of the model are $\beta \in [0, \infty)$ the repulsion intensity between the monomers and the solvent around them—or, equivalently, the attraction intensity in between monomers—and $\delta \in [0, \infty)$ the interaction intensity between the monomers and the hard wall. We discuss in detail the phase diagram of the model in Section 2.2 below, but let us mention already that the phase diagram is divided into two main phases:

- \mathcal{E} : an Extended phase inside which a typical trajectory has a macroscopic horizontal extension, comparable to its total length,
- \mathcal{C} : a Collapsed phase inside which the vertical width and the horizontal extension of a typical configuration are comparable, and of smaller order than its total size.

It turns out that \mathcal{E} can be divided into two sub-phases. A critical curve is indeed conjectured to partition \mathcal{E} into a Desorbed-Extended phase (\mathcal{DE}) inside which the polymer wanders away

from the hard wall and an Adsorbed-Extended phase (\mathcal{AE}) inside which the polymer is localized along the wall (see, e.g., [9], Figure 2). The situation is more subtle in the Collapsed phase where typical configurations look roughly like a globule (see Figure 2 (B)). The number of contacts between this globule and the hard-wall changes drastically inside \mathcal{C} along some other critical curve which triggers what physicists call a *surface transition*, that is a loss of analyticity of the second order term (which is called *surface order term* in the rest of the paper) of the exponential development of the partition function, whereas the leading order term (i.e., the free energy) remains linear.

The aim of our paper is to investigate the collapsed phase and in particular the surface transition mentioned above. However, due to major technical difficulties that we discuss in the first open problems of Section 3.4, we are unable to characterize rigorously this surface transition for the “full model”. For this reason, we introduce in Section 3 a simplified version of our model called the *single-bead model*. In a few words (see Section 3.1 for more details) every trajectory considered in our model can be decomposed into a family of sub-trajectories called beads. Those beads are typically of finite size in \mathcal{E} but are much larger inside \mathcal{C} . Note that, we can reasonably conjecture that inside \mathcal{C} , a typical trajectory is made of a unique macroscopic bead (this is proven, e.g., for the 2-dimensional IPDSAW in [20]). Thus, we restrict the set of allowed paths to those forming only one bead. This alternative version of the model turns out to be more tractable and should share many features with its unrestricted counterpart.

Let us give a short outline of the paper. In Section 2 below we begin with a rigorous definition of the model and then we provide a qualitative description of its phase diagram. With Theorem 2.2 we identify rigorously the Collapsed phase (\mathcal{C}) and the Extended phase (\mathcal{E}). Section 3 is dedicated to the definition of the single-bead version of the model. Theorem 3.2, which is the most important result of the paper, is stated in Section 3.3 and allows us to characterize the surface transition with the help of sharp asymptotic developments of the partition function inside \mathcal{C} . We prove Theorem 3.2 (along with Corollary 3.3) in Section 4. We delay the proofs of Theorem 2.2 and Proposition 3.1 to Section 5 for they are quite standard (apart from the random walk representation introduced in Section 4). We then collect the proofs of technical estimates in Section 6. Appendix A provides well-known results on the wetting model, and Appendix B displays a (conditional) FKG inequality on random walks with distribution \mathbf{P}_β (defined in (2.8) below).

2. Description of the model and phase diagram.

2.1. *The model.* For a polymer of length $L \in \mathbb{N}$, the set of its allowed configurations is denoted by Ω_L^+ and consists of those trajectories of a $1 + 1$ -dimensional self-avoiding random walk on \mathbb{Z}^2 taking unitary steps up, down and to the right and constrained to remain above the horizontal axis $y = 0$. An alternative representation of such trajectories can be given by decomposing them according to their number of horizontal/rightward steps, and the length and orientation of the vertical stretches in between, that is,

$$(2.1) \quad \Omega_L^+ := \bigcup_{N \geq 1} \mathcal{L}_{N,L}^+ := \bigcup_{N \geq 1} \left\{ (\ell_i)_{i=1}^N \in \mathbb{Z}^N; \sum_{i=1}^N |\ell_i| = L - N, \sum_{i=1}^k \ell_i \geq 0, \forall k \leq N \right\}.$$

Henceforth, we only use this latter representation and we note that each vertical stretch is followed by a horizontal step—in particular we assume that all trajectories end with a horizontal step. For every $\ell \in \Omega_L^+$, we denote by N_ℓ its horizontal extension (i.e., its number of horizontal steps) so that $\ell \in \mathcal{L}_{N_\ell,L}^+$.

With each configuration we associate a Hamiltonian, which takes into account that monomers are both attracting each other and adsorbed along the x -axis. To be more specific, a given $\ell \in \Omega_L^+$ is assigned an energetic reward $\beta \geq 0$ for every *self-touching* (i.e., a

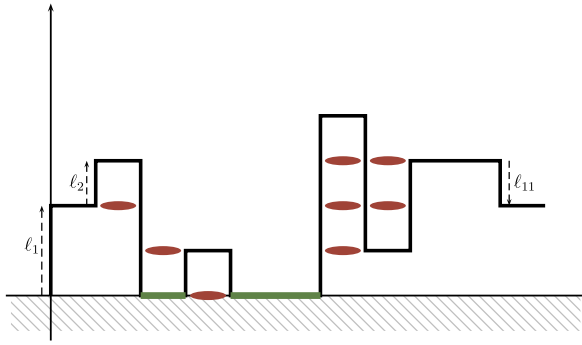


FIG. 1. Representation of a polymer of length $L = 29$, with horizontal extension $N = 11$. Each self-touching (in red) is rewarded with an energy β , and each contact with the wall (in green) is rewarded with δ .

pair of neighboring sites visited nonconsecutively by ℓ) and an energetic reward $\delta \geq 0$ for every contact with the x -axis (see Figure 1). Thus, define

$$(2.2) \quad H(\ell) := \beta \sum_{i=0}^{N_\ell} \ell_i \tilde{\wedge} \ell_{i+1} + \delta \sum_{k=1}^{N_\ell} \mathbf{1}_{\{\sum_{i=1}^k \ell_i = 0\}},$$

where the operator $\tilde{\wedge}$ is defined for any $x, y \in \mathbb{Z}$ by $x \tilde{\wedge} y := \min\{|x|, |y|\} \mathbf{1}_{\{xy \leq 0\}}$, and where we set $\ell_0 = \ell_{N_\ell+1} = 0$ for notational convenience. At this stage, we introduce the polymer measure $\mathbf{P}_L^{\beta, \delta}$, a probability on Ω_L^+ defined as

$$(2.3) \quad \mathbf{P}_L^{\beta, \delta}(\ell) = \frac{e^{H(\ell)}}{Z_{L, \beta, \delta}^+}, \quad \ell \in \Omega_L^+,$$

where $Z_{L, \beta, \delta}^+$ is a normalization term called *partition function* of the system. The *free energy* provides the exponential growth rate of $Z_{L, \beta, \delta}^+$ in L . It is defined as

$$(2.4) \quad f(\beta, \delta) = \lim_{L \rightarrow \infty} \frac{1}{L} \log Z_{L, \beta, \delta}^+,$$

(we show that f is well defined in the proof of Theorem 2.2).

REMARK 2.1. The present model may be seen as an advanced version of IPDSAW, that was introduced in [30]. For the latter model, there is no hard-wall preventing the polymer to enter the lower half-plane and also no wetting interaction with the hard-wall. As a consequence, the allowed configurations of IPDSAW are obtained by relaxing the constraint $\sum_{i=1}^k \ell_i \geq 0$ in (2.1) and the Hamiltonian by removing the term $\delta \sum_{k=1}^N \mathbf{1}_{\{\sum_{i=1}^k \ell_i = 0\}}$ in (2.2).

2.2. *Phase diagram.* Since the coupling parameters β and δ are both nonnegative, the phase diagram is drawn on the first quadrant $\mathcal{Q} := [0, \infty)^2$. Similarly to what is observed for IPDSAW (see, e.g., [23] or [5]), the phase diagram can be divided into a collapsed phase \mathcal{C} inside which typical trajectories undergo a self-touching saturation and an extended phase \mathcal{E} inside which the horizontal extension of a typical trajectory is comparable to its total size. To be more specific, inside \mathcal{C} , we expect that a typical trajectory of length L (i.e., ℓ sampled from $\mathbf{P}_L^{\beta, \delta}$) satisfies $H(\ell) = \beta L + o(L)$. For this reason its horizontal extension must be small (i.e., $N_\ell = o(L)$) and its vertical stretches should be long with alternating signs. In \mathcal{E} , in turn, a typical trajectory is expected to be composed of $O(L)$ vertical stretches of finite length (which is also what is expected at $\beta = 0$).

Let us now briefly explain (with three simple observations) why the free energy $f(\beta, \delta)$ is equal to β in \mathcal{C} . First, $f(\beta, \delta) \geq \beta$ for every $(\delta, \beta) \in \mathcal{Q}$. This inequality derives from restricting the computation of $Z_{L,\beta,\delta}^+$ to a unique trajectory $\tilde{\ell} \in \mathcal{L}_{\sqrt{L},L}$ given by

$$(2.5) \quad \tilde{\ell}_i = (-1)^{i-1}(\sqrt{L} - 1) \quad \text{for every } i \in \{1, \dots, \sqrt{L}\},$$

(we assume $\sqrt{L} \in \mathbb{N}$ for conciseness) so that $H(\tilde{\ell}) = \beta(\sqrt{L} - 1)^2 + \delta(\sqrt{L} - 1)$. Second, those trajectories in Ω_L that are performing a self-touching saturation are not many and therefore they do not carry any entropy. Third, we mentioned above that such saturated trajectories are made of $o(L)$ stretches and a trajectory may touch the hard-wall at most once per vertical stretch (recall Figure 1), hence their interactions with the wall cannot be numerous enough to contribute to the free energy. These three points are sufficient to understand why the free energy equals β in \mathcal{C} and thus, it is natural to define the *excess* free energy of the system as

$$(2.6) \quad \tilde{f}(\beta, \delta) = f(\beta, \delta) - \beta,$$

which allows us to define the *extended* and the *collapsed* phases as

$$(2.7) \quad \begin{aligned} \mathcal{C} &:= \{(\beta, \delta) \in \mathcal{Q} : \tilde{f}(\beta, \delta) = 0\}, \\ \mathcal{E} &:= \mathcal{Q} \setminus \mathcal{C} = \{(\beta, \delta) \in \mathcal{Q} : \tilde{f}(\beta, \delta) > 0\}. \end{aligned}$$

Before stating Theorem 2.2 below, we need to settle some notation. We define for any $\beta > 0$ the following probability distribution on \mathbb{Z} :

$$(2.8) \quad \mathbf{P}_\beta(\cdot = k) = \frac{e^{-\frac{\beta}{2}|k|}}{c_\beta}, \quad c_\beta := \sum_{k \in \mathbb{Z}} e^{-\frac{\beta}{2}|k|} = \frac{1 + e^{-\beta/2}}{1 - e^{-\beta/2}}.$$

For $z \in \mathbb{Z}$, we denote by $\mathbf{P}_{\beta,z}$ and $\mathbf{E}_{\beta,z}$ the law and the associated expectation of a one-dimensional random walk $X := (X_i)_{i \geq 0}$ starting from z (i.e., $\mathbf{P}_{\beta,z}(X_0 = z) = 1$) and such that $(X_{i+1} - X_i)_{i \geq 0}$ is an i.i.d. sequence of random variables with law \mathbf{P}_β . Then, we let $h_\beta(\delta)$ be the free energy of the wetting model that consists of the random walk X constrained to remain nonnegative and to finish on the x -axis, and pinned at the origin by an energetic factor δ , that is,

$$(2.9) \quad h_\beta(\delta) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbf{E}_{\beta,0} [e^{\delta \sum_{i=1}^N 1_{\{X_i=0\}}} 1_{\{X \in B_N^{0,+}\}}],$$

where $B_N^{0,+}$ is the set of nonnegative trajectories of length N ending at 0—more generally, define for all $y \geq 0$,

$$(2.10) \quad B_N^{y,+} := \{(X_k)_{k=1}^N \in \mathbb{Z}^N; X_N = y, X_k \geq 0 \forall 1 \leq k \leq N\}.$$

An explicit formula for $h_\beta(\delta)$ is given in Appendix A (see (A.4)). We also define $\Gamma_\beta := c_\beta e^{-\beta}$ which is decreasing in β , and $\beta_c > 0$ the unique solution of the equation $\Gamma_\beta = 1$.

THEOREM 2.2. *The boundary between the collapsed and the extended phase can be characterized explicitly, that is,*

$$\mathcal{C} = \{(\beta, \delta) \in \mathcal{Q} : \beta \geq \beta_c, \delta \leq \delta_c(\beta)\},$$

where, for every $\beta \geq \beta_c$, the quantity $\delta_c(\beta)$ is the unique solution in δ of

$$(2.11) \quad \log \Gamma_\beta + h_\beta(\delta) = 0,$$

which yields the following analytic expression:

$$(2.12) \quad \delta_c(\beta) = \log \left(\frac{\sinh(\beta) + \sqrt{\sinh(\beta)^2 + 1 - e^\beta}}{1 - e^{-\beta}} \right).$$

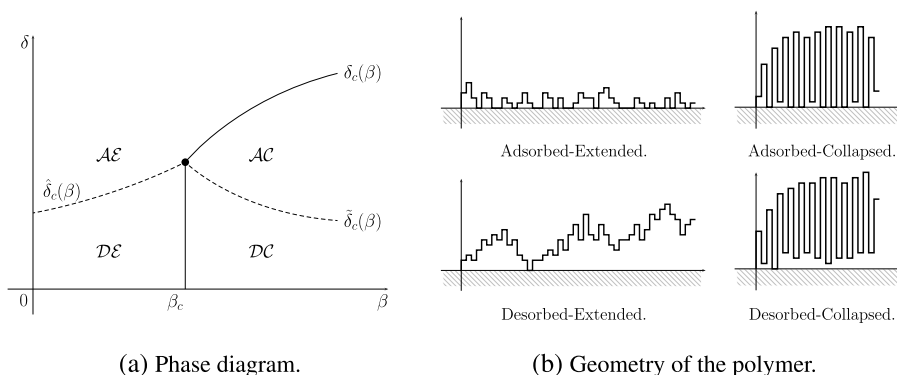


FIG. 2. Qualitative picture of the phase diagram and the geometry of the polymer in each phase. In this paper, we rigorously determine the critical line $\delta_c(\beta)$ between the extended and collapsed phases, and we characterize the surface transition critical line $\tilde{\delta}_c(\beta)$. No analytic expression has been conjectured for the critical line $\tilde{\delta}_c(\beta)$ yet.

REMARK 2.3. Note that our formula for the critical curve in (2.12) was already conjectured in [10], Equation 19, or [14], Equation 21, (both expressions coincide provided we set $\kappa = e^{\delta_c(\beta)}$ and $\tau = e^\beta$). In [10] and [14], the heuristics supporting this formula are based on some additional assumption and on a computation of the grand canonical partition function with the help of a transfer matrix.

2.2.1. Discussion. Let us further explain the phenomenon behind the existence of a surface transition inside \mathcal{C} that physicists have conjectured (see, e.g., [21], Figure 2). As mentioned above, a typical trajectory in the collapsed phase looks like a globule delimited by a lower envelope and an upper envelope (see their rigorous definition in (3.20)). For $L \in \mathbb{N}$, $\beta > \beta_c$ and $\delta < \delta_c(\beta)$, we prove in Section 4.2 that the lower envelope of a trajectory sampled from $\mathbf{P}_L^{\beta, \delta}$ behaves roughly as a random walk of length $O(\sqrt{L})$, constrained to remain nonnegative, and pinned at the x -axis (hard wall) with intensity δ . This leaves us with a wetting model whose critical point $\tilde{\delta}_c(\beta)$ can be explicitly computed, see (3.7) (and it satisfies $\tilde{\delta}_c(\beta) < \delta_c(\beta)$). Thus, when $\delta \leq \tilde{\delta}_c(\beta)$ the lower-envelope touches the hard-wall only $o(\sqrt{L})$ times, whereas when $\delta > \tilde{\delta}_c(\beta)$ it remains localized along the hard wall and touches it $O(\sqrt{L})$ times. As a consequence, this wetting transition of the lower envelope is not encoded in the excess free energy $\tilde{f}(\beta, \delta)$ (which remains equal to 0 in \mathcal{C}) simply because the number of contacts between the polymer (or equivalently its lower envelope) and the hard-wall is at most $O(\sqrt{L})$. To be more specific, we claim in Theorem 3.2 below that, in the exponential growth rate of the partition function (for the single-bead model, see Section 3.2), $\delta \in (\tilde{\delta}_c(\beta), \delta_c(\beta))$ only contributes to the surface order term. We further discuss the behavior of a typical lower envelope in \mathcal{C} below the statement of Theorem 3.2.

In order to display a qualitative picture of the phase diagram (see Figure 2), let us end this discussion with a few words about the extended regime \mathcal{E} , where a typical trajectory of length L is expected to have an horizontal extension of order L . Physicists (see [10] or [14], Figure 2) have conjectured that another critical curve $\hat{\delta}_c : [0, \beta_c] \rightarrow \mathbb{R}^+$ divides \mathcal{E} into a Desorbed-Extended phase denoted by $\mathcal{D}\mathcal{E}$ and an Adsorbed-Extended phase denoted by $\mathcal{A}\mathcal{E}$ but there is so far no guess for what the value of $\hat{\delta}_c(\beta)$ could be.

3. Inside the collapsed phase: Restriction to the single-bead model. Asymptotics of the partition functions.

3.1. Bead decomposition of a trajectory. A trajectory of Ω_L^\dagger can be decomposed into a collection of sub-trajectories called beads. A bead is a succession of nonzero vertical stretches

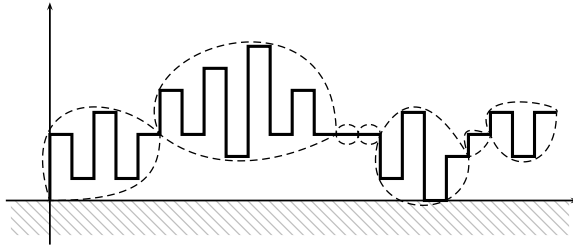


FIG. 3. Decomposition of a trajectory into a succession of seven beads. For all $\ell \in \Omega_L^+$ and $1 \leq i \leq N_\ell$, a new bead starts at the i th vertical stretch if and only if $\ell_{i-1}\ell_i \geq 0$ (where we recall $\ell_0 = \ell_{N_\ell+1} = 0$).

with alternating signs, which ends when two consecutive stretches have the same orientation, or when a stretch has length zero (see Figure 3). To be more specific, we consider $\ell \in \Omega_L^+$ and we recall that N_ℓ is its horizontal extension and that by convention $\ell_0 = \ell_{N_\ell+1} = 0$. Then, we set $x_0 = 0$ and for $j \in \mathbb{N}$ such that $x_{j-1} < N_\ell$ we set

$$x_j = \inf\{i \geq x_{j-1} + 1 : \ell_i \tilde{\wedge} \ell_{i+1} = 0\},$$

so that x_j is the index of the last vertical stretch composing the j th bead of ℓ . Finally, we let $n(\ell)$ be the number of beads in ℓ , in particular it satisfies $x_{n(\ell)} = N_\ell$. Thus, we can decompose any trajectory $\ell \in \Omega_L$ into a succession of beads denoted by \mathcal{B}_j with $j \in \{1, \dots, n(\ell)\}$ as follows:

$$(3.1) \quad \ell = \bigodot_{j=1}^{n(\ell)} \mathcal{B}_j := \bigodot_{j=1}^{n(\ell)} (\ell_{x_{j-1}+1}, \dots, \ell_{x_j}),$$

where \odot denotes the concatenation.

A key issue concerning the collapsed phase of our model consists in showing that a typical trajectory contains a unique macroscopic bead outside which very few monomers are laying. Such result was derived for IPDSAW in [4] and recently improved in [20]. Its proof requires some sharp asymptotics of the partition function restricted to beads. Those estimates are a lot more technical in the model with a hard wall, since the shape of a bead strongly depends on its position with respect to the wall (i.e., its starting and ending heights). In the present paper, we consider the model restricted to describe a single-bead which begins and ends at the wall, see Section 3.2 below for a definition. Not only this restriction makes the model more tractable, we also conjecture that the multiple-bead model, inside its collapsed phase, describes a unique macroscopic bead of the same nature as the one studied in this paper, that is, beginning and ending in the vicinity of the wall.

3.2. *Single-bead restriction of the model.* Let $L \in \mathbb{N}$, and define Ω_L^{bead} the subset of Ω_L^+ gathering those trajectories ℓ constrained to form only one “bead”—all its stretches are of nonzero length and alternate orientations—and to come back to the wall with its last stretch (in particular its horizontal extension N_ℓ must be even). That is,

$$(3.2) \quad \Omega_L^{\text{bead}} := \bigcup_{N \geq 1} \mathcal{L}_{N,L}^{\text{bead}} := \bigcup_{N \geq 1} \left\{ \begin{array}{l} (\ell_i)_{i=1}^{2N} \in \mathbb{Z}^{2N}; \sum_{i=1}^{2N} |\ell_i| = L - 2N, \sum_{i=1}^{2N} \ell_i = 0, \\ \sum_{i=1}^k \ell_i \geq 0, \forall k \leq 2N, \ell_i \ell_{i+1} < 0, \forall 1 \leq i < 2N \end{array} \right\}.$$

The partition function restricted to such trajectories becomes

$$(3.3) \quad Z_{L,\beta,\delta}^{\text{bead}} := \sum_{N=1}^{L/2} \sum_{\substack{\ell \in \mathcal{L}_{N,L}^{\text{bead}} \\ \ell_0 = \ell_{2N+1} = 0}} e^{H(\ell)},$$

(recall that $\ell_0 = \ell_{2N+1} = 0$ for notational convenience). We recall (2.3) and denote by $\mathbf{P}_{L,\beta,\delta}^{\text{bead}}$ the law $\mathbf{P}_L^{\beta,\delta}$ conditioned on $\ell \in \Omega_L^{\text{bead}}$, that is,

$$(3.4) \quad \mathbf{P}_{L,\beta,\delta}^{\text{bead}}(\ell) := \mathbf{P}_L^{\beta,\delta}(\ell | \ell \in \Omega_L^{\text{bead}}) = \frac{e^{H(\ell)}}{Z_{L,\beta,\delta}^{\text{bead}}}, \quad \ell \in \Omega_L^{\text{bead}}.$$

3.3. *Surface transition: Asymptotics of the single-bead partition function inside the collapsed phase.* The single-bead model undergoes the same phase transition as the full model, albeit its critical curve differs slightly. Let us define f^{bead} and \tilde{f}^{bead} respectively, the free energy and excess free energy of the single-bead model,

$$f^{\text{bead}}(\beta, \delta) := \lim_{L \rightarrow \infty} \log Z_{L,\beta,\delta}^{\text{bead}}, \quad \text{and} \quad \tilde{f}^{\text{bead}}(\beta, \delta) := f^{\text{bead}}(\beta, \delta) - \beta.$$

Since $\tilde{\ell}$ forms a single bead (recall (2.5)), it follows that $\tilde{f}^{\text{bead}}(\beta, \delta) \geq 0$ for all $(\beta, \delta) \in \mathcal{Q}$.

PROPOSITION 3.1. *For the single-bead model, we have*

$$(3.5) \quad \mathcal{C}_{\text{bead}} := \{(\beta, \delta) \in \mathcal{Q} : \tilde{f}^{\text{bead}}(\beta, \delta) = 0\} = \{(\beta, \delta) \in \mathcal{Q} : \beta \geq \beta_c, \delta \leq \delta_c^{\text{bead}}(\beta)\},$$

where, for every $\beta \geq \beta_c$, the quantity $\delta_c^{\text{bead}}(\beta)$ is the unique solution in δ of

$$(3.6) \quad 2 \log \Gamma_\beta + h_\beta(\delta) = 0.$$

Notice that an analytic expression of $\delta_c^{\text{bead}}(\beta)$ can be derived from (3.6) and (A.4), similarly to (2.12) in Theorem 2.2. Let us now focus on the collapsed phase of the single-bead model. The surface transition occurs along a curve denoted by $\tilde{\delta}_c : [\beta_c, \infty) \mapsto \mathbb{R}^+$ where $\tilde{\delta}_c(\beta)$ turns out to be the critical point of the wetting model introduced in (2.9), that is for every $\beta \geq \beta_c$,

$$(3.7) \quad \tilde{\delta}_c(\beta) := \inf\{\delta \geq 0 : h_\beta(\delta) > 0\} = -\log(1 - e^{-\beta/2}).$$

The second identity in (3.7) is proven in Proposition A.1. Definitions (3.6) and (3.7) ensure us that $\tilde{\delta}_c(\beta) \leq \delta_c(\beta) \leq \delta_c^{\text{bead}}(\beta)$ for $\beta \geq \beta_c$: thus, the curve $\tilde{\delta}_c : [\beta_c, \infty) \mapsto \mathbb{R}^+$ lies in both \mathcal{C} and $\mathcal{C}_{\text{bead}}$. We claim in Theorem 3.2 below that the surface order term in the exponential development of $Z_{L,\beta,\delta}^{\text{bead}}$ loses its analyticity along that curve.

At this stage, we divide the collapsed phase $\mathcal{C}_{\text{bead}}$ into a *desorbed collapsed* phase \mathcal{DC} and an *adsorbed collapsed* phase \mathcal{AC} defined as

$$(3.8) \quad \begin{aligned} \mathcal{DC} &:= \{(\beta, \delta) \in \mathcal{Q} : \beta \geq \beta_c, \delta \leq \tilde{\delta}_c(\beta)\}, \\ \mathcal{AC} &:= \{(\beta, \delta) \in \mathcal{Q} : \beta \geq \beta_c, \tilde{\delta}_c(\beta) \leq \delta \leq \delta_c^{\text{bead}}(\beta)\}, \end{aligned}$$

where we dropped the subscript “bead” to lighten notation. To fully state Theorem 3.2, we need to introduce some definitions. Let \mathcal{L} be the logarithmic moment generating function of the distribution \mathbf{P}_β (recall (2.8)), that is for any $|h| < \beta/2$,

$$(3.9) \quad \mathcal{L}(h) := \log \mathbf{E}_\beta[e^{hX_1}],$$

and for every $\mathbf{h} = (h_0, h_1) \in \mathcal{D}_\beta := \{(h_0, h_1) \in \mathbb{R}^2, |h_1| < \beta/2, |h_0 + h_1| < \beta/2\}$, define

$$(3.10) \quad \mathcal{L}_\Lambda(\mathbf{h}) := \int_0^1 \mathcal{L}(h_0x + h_1) dx,$$

which is strictly convex on \mathcal{D}_β . In [4], Lemma 5.3, it is proven that $\mathbf{h} \in \mathcal{D}_\beta \mapsto \nabla \mathcal{L}_\Lambda(\mathbf{h}) = (\partial_{h_0} \mathcal{L}_\Lambda, \partial_{h_1} \mathcal{L}_\Lambda)(\mathbf{h})$ is a C^1 -diffeomorphism from \mathcal{D}_β to \mathbb{R}^2 , so let $\tilde{\mathbf{h}} : \mathbb{R}^2 \rightarrow \mathcal{D}_\beta$ be its inverse and from now on we note $\tilde{\mathbf{h}} := \tilde{\mathbf{h}}(\cdot, \cdot)$ to tighten notation.

Define $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ the Legendre transform of \mathcal{L}_Λ , that is,

$$(3.11) \quad g(q, p) := \tilde{\mathbf{h}}^{q,p} \cdot (q, p) - \mathcal{L}_\Lambda(\tilde{\mathbf{h}}^{q,p}).$$

Since \mathcal{L}_Λ is convex, g is convex as well (see [28], Theorem 12.2). It is also nonnegative and a straightforward differentiation ensures us that

$$(3.12) \quad \nabla g(q, p) := (\partial_q g, \partial_p g)(q, p) = \tilde{\mathbf{h}}^{q,p},$$

which yields that g is C^2 and strictly convex. With those notations at hand, define

$$(3.13) \quad \varphi_{(\beta,\delta)}(a) := a \left(2 \log \Gamma_\beta + h_\beta(\delta) - g\left(\frac{1}{2a^2}, 0\right) \right), \quad a \in (0, \infty).$$

For $\beta > \beta_c$ and $\delta < \delta_c^{\text{bead}}(\beta)$, the quantity $2 \log \Gamma_\beta + h_\beta(\delta)$ is negative. Therefore, $\varphi_{(\beta,\delta)}$ is negative on $(0, \infty)$ and its first two derivatives are computed as

$$\begin{aligned} \varphi'_{(\beta,\delta)}(a) &= 2 \log \Gamma_\beta + h_\beta(\delta) - g\left(\frac{1}{2a^2}, 0\right) + \frac{1}{a^2} \partial_q g\left(\frac{1}{2a^2}, 0\right), \\ \varphi''_{(\beta,\delta)}(a) &= -\frac{1}{a^3} \partial_q g\left(\frac{1}{2a^2}, 0\right) - \frac{1}{a^5} \partial_q^2 g\left(\frac{1}{2a^2}, 0\right). \end{aligned}$$

At this stage, the strict convexity of g combined with the fact that $\nabla g(0, 0) = (0, 0)$ (by (3.12)) implies that $\varphi_{(\beta,\delta)}$ is strictly concave (since $\varphi''_{(\beta,\delta)} < 0$ on $(0, \infty)$) and that $\varphi'_{(\beta,\delta)}(a) < 0$ for a large enough. It remains to write

$$(3.14) \quad \varphi'(a) = 2 \log \Gamma_\beta + h_\beta(\delta) + \frac{1}{2a^2} \tilde{h}_0^{\frac{1}{2a^2}, 0} + \mathcal{L}_\Lambda(\tilde{\mathbf{h}}^{\frac{1}{2a^2}, 0}),$$

and to note that $\lim_{a \searrow 0} \tilde{h}_0^{\frac{1}{2a^2}, 0} = \frac{\beta}{2} > 0$ and that $\mathcal{L}_\Lambda(\tilde{\mathbf{h}}^{\frac{1}{2a^2}, 0}) \geq 0$; hence $\varphi'(a) > 0$ for a sufficiently small. This proves that $\sup_{(0, \infty)} \varphi = \varphi(\tilde{a})$ for some $\tilde{a} = \tilde{a}(\beta, \delta) \in (0, \infty)$.

Finally, set $\Phi(\beta, \delta) := \varphi_{(\beta,\delta)}(\tilde{a}(\beta, \delta))$, let σ_β^2 be the variance of X_1 under \mathbf{P}_β , and recall (0.2) for the definition of \asymp . Recall also (3.6) and (3.7) for the definitions of $\delta_c^{\text{bead}}(\beta)$ and $\tilde{\delta}_c(\beta)$, respectively. We now claim our main result.

THEOREM 3.2. *Let $\beta > \beta_c$.*

(i) *For $\delta \in (\tilde{\delta}_c(\beta), \delta_c^{\text{bead}}(\beta))$, then*

$$(3.15) \quad Z_{L,\beta,\delta}^{\text{bead}} \asymp \frac{1}{L^{3/4}} e^{\beta L + \Phi(\beta,\delta)\sqrt{L}}.$$

(ii) *For $\delta \in (0, \tilde{\delta}_c(\beta)]$ and $\varepsilon > 0$, there exist $C > 0$ and $L_0 \in \mathbb{N}$ such that for $L \geq L_0$,*

$$(3.16) \quad e^{\beta L + \Phi(\beta,0)\sqrt{L} + (\Psi(\beta) - \varepsilon)L^{1/6}} \leq Z_{L,\beta,\delta}^{\text{bead}} \leq \frac{C}{L^{3/4}} e^{\beta L + \Phi(\beta,0)\sqrt{L}}.$$

(iii) *For $\delta = 0$, then*

$$(3.17) \quad Z_{L,\beta,\delta}^{\text{bead}} = e^{\beta L + \Phi(\beta,0)\sqrt{L} + \Psi(\beta)L^{1/6(1+o(1))}}, \quad \text{as } L \rightarrow \infty,$$

where

$$\Psi(\beta) := -|a_1| \left[\frac{\tilde{a}(\beta, 0)\sigma_\beta^2}{2} \tilde{h}_0 \left(\frac{1}{2\tilde{a}(\beta, 0)}, 0 \right)^2 \right]^{1/3},$$

and a_1 denotes the first zero (in absolute value) of the Airy function.

These estimate allow us to derive some properties of typical trajectories under the polymer measure, most notably regarding the number of contacts with the hard wall in the collapsed phase.

COROLLARY 3.3. *Let $\beta > \beta_c$.*

(i) *The function $\delta \mapsto \Phi(\beta, \delta)$ is C^1 on $(\tilde{\delta}_c(\beta), \delta_c^{\text{bead}}(\beta))$. For $\delta \in (\tilde{\delta}_c(\beta), \delta_c^{\text{bead}}(\beta))$ and for any $\varepsilon > 0$ we have*

$$(3.18) \quad \lim_{L \rightarrow \infty} \mathbf{P}_{L, \beta, \delta}^{\text{bead}} \left(\sum_{k=1}^{N_\ell} \mathbf{1}_{\{\sum_{i=1}^k \ell_i = 0\}} \in [\partial_\delta \Phi(\beta, \delta) - \varepsilon, \partial_\delta \Phi(\beta, \delta) + \varepsilon] \sqrt{L} \right) = 1.$$

(ii) *For $\delta \in [0, \tilde{\delta}_c(\beta))$, there exist some $K > 0$ (which only depends on β) such that*

$$(3.19) \quad \lim_{L \rightarrow \infty} \mathbf{P}_{L, \beta, \delta}^{\text{bead}} \left(\sum_{k=1}^{N_\ell} \mathbf{1}_{\{\sum_{i=1}^k \ell_i = 0\}} \leq KL^{1/6} \right) = 1.$$

REMARK 3.4. • The surface transition occurring along the curve $\{(\beta, \tilde{\delta}_c(\beta)), \beta > \beta_c\}$ is proven by Theorem 3.2(i)–(ii) (and confirmed by Corollary 3.3). Indeed, $\delta > \tilde{\delta}_c(\beta)$ implies that $h_\beta(\delta) > h_\beta(0) = 0$, hence $\varphi_{(\beta, \delta)}(a) > \varphi_{(\beta, 0)}(a)$ for $a \in (0, \infty)$, and $\Phi(\beta, \delta) > \Phi(\beta, 0)$ (whereas $h_\beta(\delta) = 0$ for all $\delta \leq \tilde{\delta}_c(\beta)$).

• In Theorem 3.2(ii), we conjecture that the upper bound is not optimal, and that (iii) should apply at least for every $\delta < \tilde{\delta}_c(\beta)$. Similarly, for Corollary 3.3(ii), we expect the typical number of contacts with the hard wall to be of smaller order than $L^{1/6}$.

• In Theorem 3.2(iii), obtaining $\Psi(\beta)$ requires to compute the Laplace transform of the area enclosed by a Brownian meander of length T . The first zero of the Airy function a_1 appears in the leading order of such Laplace transform as $T \rightarrow \infty$ (see Section 4.4).

3.3.1. *Discussion.* Let us give some insights into Theorem 3.2. For any trajectory $\ell \in \Omega_L^{\text{bead}}$ forming a single bead, we define its *lower envelope* $I := (I_i)_{i=0}^{N_\ell/2}$ and *upper envelope* $S := (S_i)_{i=0}^{N_\ell/2+1}$ as follow:

$$(3.20) \quad S_k = \sum_{i=1}^{2k-1} \ell_i, \quad k \in \left\{ 1, \dots, \frac{N_\ell}{2} + 1 \right\},$$

$$I_k = \sum_{i=1}^{2k} \ell_i, \quad k \in \left\{ 1, \dots, \frac{N_\ell}{2} \right\},$$

and $S_0 = I_0 = 0$. The single-bead constraint ensures that S (resp. I) describes the topmost (resp. bottommost) layer of the polymer (see Figure 4).

In Section 4.1 below we show that, under the polymer measure, the envelopes I and S of a given single-bead configuration may be sampled as trajectories of nonnegative random walk bridges that are coupled via geometric constraints. In Section 4.2 we break that geometric coupling and integrate over S so that I can be investigated on its own (see (4.10)). However, this comes with a cost (see Proposition 4.2 below), the law of I being perturbed by:

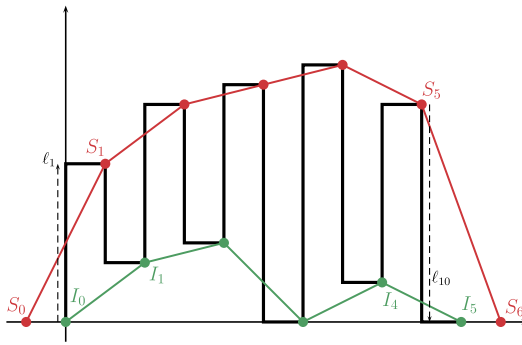


FIG. 4. Representation of a single-bead trajectory, with its upper envelope (in red) and its lower envelope (in green) respectively described by the sequences S and I .

1. a wetting term $\delta \sum_{k=1}^N 1_{\{I_k=0\}}$ which comes from the fact that the polymer (or equivalently its lower envelope) is adsorbed along the hard wall,
2. a pre-wetting term proportional to $-A_N(I)/N$ where $A_N(I)$ is the area below I (see (4.6)). The latter penalization comes from the fact that the upper envelope S must sweep an abnormally large area, that is, $A_N(S) = A_N(I) + qN^2$ with $q > 0$. Therefore, S , which is already in a large deviation regime, pushes down the lower envelope I so as to keep $A_N(I)$ small.

The influence of a pre-wetting term on a $1 + 1$ -dimensional random walk constrained to remain positive has already been studied in [13] and [15] (see also [16] for a review). Among physical motivations is, for example, the study of a liquid-gas interface when a thermodynamically stable gas is in contact with a substrate (hard-wall) that has a strong preference for the liquid phase, with the temperature decreasing to the liquid/gas critical point. From this point of view, the present paper displays a new example of a physical object (i.e., the lower envelope of a collapsed homopolymer interacting with a hard wall) associated with a model for which pre-wetting appears naturally.

As stated in Theorem 3.2, the pre-wetting term does not have an influence on the lower-envelope inside \mathcal{AC} since the pinning term is strong enough to keep the lower envelope at finite distance from the hard-wall. Inside \mathcal{DC} , in turn the pre-wetting term should dominate and we expect that [13], Theorem 1.2, also applies here, implying that I has fluctuations of order $L^{1/6}$ (its length being \sqrt{L}).

3.4. *Open problems.* From a mathematical point of view, there are still many issues that remain to be settled concerning the present model. Let us list a few here.

1. Consider the multiple-bead model (defined in (2.1)–(2.4)) and prove that, inside its collapsed phase \mathcal{C} , it undergoes the same surface transition as the single-bead model. We conjecture that it takes place along the very same critical curve $\beta \mapsto \tilde{\delta}_c(\beta)$ than in the single-bead model (even though the phases \mathcal{C} and $\mathcal{C}_{\text{bead}}$ differ slightly). However, this is actually a very intricate matter to investigate, since the bead decomposition requires us to consider the heights of starting and ending points of each bead in addition to their length. Moreover, our estimates for the single-bead model rely noticeably on the observation that the area enclosed by the lower envelope is not too large ($A_N(I) = O(N^{3/2})$), since the bead is close to the wall. This observation fails for beads away from the wall, for instance when the height of their starting and/or ending point are of order N . For such beads the pressure applied by the upper envelope should push the lower envelope from at most standard fluctuations into a large deviation trajectory (around a convex profile). Furthermore, the interaction between the

wall and the lower envelope seems very intricate to study, even regarding the critical value of the adsorption transition. Thus, the variational problem (3.13) for those beads—hence the surface term in the logarithmic development of their partition function—should differ from beads close to the wall (even though trajectories involving faraway beads are expected to be unlikely in the collapsed multiple-bead model).

2. For the multiple-bead model again, prove that a typical trajectory is made of a unique macroscopic bead, and give a bound on the number of monomers that may lay outside this bead. A similar result was proven in [4], Theorem C, and [20] for the model without wall, and comforts us in the idea that, in the Collapsed phase, the single-bead model encapsulates quite accurately the behavior of its multiple-bead counterpart.

3. Consider a random walk $(X_i)_{i=0}^N$ of law $\mathbf{P}_{\beta,0}$ (recall (2.8)) constrained to remain nonnegative and perturbed by both a wetting and a pre-wetting terms of parameter δ and γ respectively, that is,

$$(3.21) \quad \mathbf{E}_{\beta,0} \left[e^{\delta \sum_{i=1}^N \mathbf{1}_{\{U_k=0\}}} \mathbf{1}_{\{I \in B_N^{0,+}\}} e^{-\frac{\gamma}{N} A_N(X)} \right].$$

Display sharp asymptotics for the partition function in (3.21) when δ is not larger than the critical point of the pure wetting model (i.e., $\delta \leq \tilde{\delta}_c(\beta)$). Such estimates would be the key to improve Theorem 3.2 and Corollary 3.3 in \mathcal{DC} . Note that a related model has been considered in the physics literature in [1] to investigate a system where two phases A and B are coexisting on top of a horizontal hard wall. Depending on the chemical affinities between phase A and phase B on the one hand and between the wall and the phases A and B on the other hand, a film of solvent B may appear between the wall and phase A . Among other results, the typical width of such film and the correlation lengths in the system are derived by using spectral tools.

4. Provide a characterization of the critical curve dividing \mathcal{E} into an Adsorbed-Extended phase and a Desorbed-Extended phase. In the Extended regime, the bead decomposition and upper/lower envelopes description are far less efficient, so a whole different approach may be needed.

4. Proof of Theorem 3.2. We divide the proof of Theorem 3.2 into six steps. First, we adapt the random-walk representation of IPDSAW initially introduced in [23] to the present single-bead model. We derive a probabilistic representation of the partition function by rewriting, for every $N \leq L/2$, the contribution to the partition function of those trajectories made of $2N$ stretches, in terms of two auxiliary random walks S and I . One particularity comes from the fact that I and S are coupled since the area enclosed in-between S and I is imposed by the length of the polymer, and another one comes from the single-bead constraint which implies that they cannot cross trajectories: hence S (resp. I) plays the role of the *upper envelope* (resp. *lower envelope*) of the polymer. The second step consists in breaking the geometric coupling between I and S . This is achieved by integrating over S , and transforming the area constraint into an exponential perturbation of the law of I . In the third and fourth steps, we estimate the partition function of the lower envelope I , in \mathcal{AC} and \mathcal{DC} respectively. Finally, the fifth step proves that inside the collapsed phase, the horizontal extension of a typical trajectory is of order \sqrt{L} , and the sixth step collects all those estimates to prove Theorem 3.2.

4.1. *Step 1: Random-walk representation.* The understanding of IPDSAW (recall Remark 2.1) has recently been improved (see [5] for a review). The key tool was a new probabilistic representation of the partition function based on an auxiliary random walk conditioned to enclose a prescribed area. It turns out that this representation is of substantial help for the present model as well but under a different form. This is the object of the present step.

Let us first provide a probabilistic description of the single-bead partition function $Z_{L,\beta,\delta}^{\text{bead}}$. We recall (2.2) and (3.3) and we observe that

$$(4.1) \quad x \tilde{\wedge} y = \frac{|x| + |y| - |x + y|}{2}, \quad x, y \in \mathbb{Z}.$$

Hence the single-bead partition function can be written as

$$(4.2) \quad Z_{L,\beta,\delta}^{\text{bead}} = e^{\beta L} \sum_{N=1}^{L/2} e^{-2\beta N} \sum_{\substack{\ell \in \mathcal{L}_{N,L}^{\text{bead}} \\ \ell_0 = \ell_{2N+1} = 0}} e^{-\frac{\beta}{2} \sum_{i=0}^{2N} |\ell_i + \ell_{i+1}|} e^{\delta \sum_{k=1}^{2N} \mathbf{1}_{\{\sum_{i=1}^k \ell_i = 0\}}}.$$

At this stage, we recall (2.8) and we consider two independent random walks $S := (S_i)_{i \geq 0}$ and $I := (I_i)_{i \geq 0}$ of law $\mathbf{P}_{\beta,0}$. We notice that for every $\ell \in \mathcal{L}_{N,L}^{\text{bead}}$ (with $\ell_0 = \ell_{2N+1} = 0$) the first factor in the second sum in (4.2) satisfies

$$(4.3) \quad \begin{aligned} & e^{-\frac{\beta}{2} \sum_{i=0}^{2N} |\ell_i + \ell_{i+1}|} \\ &= c_{\beta}^{2N+1} \mathbf{P}_{\beta,0} \left(S_k = \sum_{i=0}^{2k-1} \ell_i, \forall k \leq N+1 \right) \mathbf{P}_{\beta,0} \left(I_k = \sum_{i=0}^{2k} \ell_i, \forall k \leq N \right). \end{aligned}$$

Let us introduce a one-to-one correspondence between trajectories $(\ell_i)_{i=0}^{2N+1} \in \mathbb{Z}^{2N+2}$ with $\ell_0 = \ell_{2N+1} = 0$, and $(S_i)_{i=0}^{N+1} \in \mathbb{Z}^{N+2}$, $(I_i)_{i=0}^N \in \mathbb{Z}^{N+1}$ with $S_0 = I_0 = 0$ and $S_{N+1} = I_N$, by letting $S_k = \sum_{i=0}^{2k-1} \ell_i$, $\forall 1 \leq k \leq N+1$ and $I_k = \sum_{i=0}^{2k} \ell_i$, $\forall 1 \leq k \leq N$. Then, constraints on $\ell \in \mathcal{L}_{N,L}^{\text{bead}}$ can be transcribed to S and I (recall (3.2)). Indeed, $\sum_{i=1}^{2N} \ell_i = 0$ is equivalent to $S_{N+1} = I_N = 0$, and $\sum_{i=1}^k \ell_i \geq 0$, $\forall k \leq 2N$ is equivalent to $S_i \geq 0$, $\forall i \leq N+1$ and $I_i \geq 0$, $\forall i \leq N$. Besides, we can write

$$(4.4) \quad \sum_{i=1}^{2N} |\ell_i| = \sum_{k=1}^N |I_k - S_k| + \sum_{k=1}^{N+1} |S_k - I_{k-1}| =: G(S, I),$$

which is the “geometric area” between S and I , and it is constrained to be equal to $L - 2N$ in $\mathcal{L}_{N,L}^{\text{bead}}$. Finally, $\ell_i \ell_{i+1} < 0$, $\forall 1 \leq i < 2N$ is equivalent to $S \succ I$ where we define

$$(4.5) \quad \{S \succ I\} := \{S_k > \max(I_k, I_{k-1}), \forall 1 \leq k \leq N\}.$$

In particular this means that in the single-bead model, S remains above I (thereby we respectively call S and I the *upper* and *lower envelopes* of the polymer—recall Figure 4), and it implies that we can rewrite the geometric area $G(S, I)$: defining the *signed area* as

$$(4.6) \quad A_n(X) := \sum_{i=0}^n X_n,$$

for any $n \in \mathbb{N}$ and $(X_k)_{k=0}^n \in \mathbb{Z}^{n+1}$, we then have that $\{S \succ I\}$ and $S_{N+1} = I_N = 0$ imply $G(S, I) = 2(A_{N+1}(S) - A_N(I))$.

Going back to (4.2) and plugging (4.3) in, those observations prove that we can rewrite the partition function as follows. Let $\mathbf{P}_{\beta,x}$ denote the law of a random walk on \mathbb{Z} starting from x with increments distributed as \mathbf{P}_{β} (henceforward, we omit the x -dependence in $\mathbf{P}_{\beta,x}$ when x is clear from context). Recall that $\Gamma_{\beta} = c_{\beta} e^{-\beta}$, and $B_N^{0,+}$ is defined in (2.10).

PROPOSITION 4.1. *For $\beta > 0$, $\delta \geq 0$ and $L \geq 1$,*

$$(4.7) \quad \tilde{Z}_{L,\beta,\delta}^{\text{bead}} := \frac{1}{c_{\beta} e^{\beta L}} Z_{L,\beta,\delta}^{\text{bead}} = \sum_{N=1}^{L/2} (\Gamma_{\beta})^{2N} D_{N,q_N}^{\text{bead}},$$

with $q_N^L := (L - 2N)/2N^2$, and for $q \in (0, \infty) \cap \frac{\mathbb{N}}{2N^2}$,

$$(4.8) \quad D_{N,q}^{\text{bead}} := \mathbf{E}_{\beta,0} [e^{\delta \sum_{k=1}^N \mathbf{1}_{\{I_k=0\}}} \mathbf{1}_{\{S \in B_{N+1}^{0,+}, I \in B_N^{0,+}\}} \mathbf{1}_{\{S > I\}} \mathbf{1}_{\{A_{N+1}(S) = A_N(I) + qN^2\}}],$$

where S, I are two independent random walks distributed as $\mathbf{P}_{\beta,0}$.

Our aim is to provide sharp estimates on the partition function $Z_{L,\beta,\delta}^{\text{bead}}$. To that purpose, we need to estimate $D_{N,q}^{\text{bead}}$ uniformly in $q \in [q_1, q_2] \cap \frac{1}{2N^2}\mathbb{N}$ for any $0 < q_1 < q_2$ fixed.

4.2. *Step 2: Integrating over S in $D_{N,q}^{\text{bead}}$.* With Proposition 4.2 below, we show that the geometric constraint imposed to I and S can be relaxed provided we introduce an exponential perturbation to the law of I , thus breaking the coupling between I and S .

Recall the definition of $g : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ the Legendre transform of \mathcal{L}_Λ in (3.11) that is \mathcal{C}^2 , strictly convex and satisfies (3.12). We claim below that g is also the rate function of a large deviation principle associated to the couple $\Lambda_N := (A_N(X)/N, X_N)$ under \mathbf{P}_β (see the proof of Proposition 4.3).

PROPOSITION 4.2. *Fix some $q_2 > q_1 > 0$. Then*

$$(4.9) \quad D_{N,q}^{\text{bead}} \asymp \frac{e^{-Ng(q,0)}}{N^2} E_{N,q}^{\text{bead}},$$

uniformly in $q \in [q_1, q_2] \cap \frac{1}{2N^2}\mathbb{N}$, where

$$(4.10) \quad E_{N,q}^{\text{bead}} := \mathbf{E}_{\beta,0} [e^{\delta \sum_{k=1}^N \mathbf{1}_{\{I_k=0\}}} \mathbf{1}_{\{I \in B_N^{0,+}\}} e^{-\partial_q g(q,0) \frac{A_N(I)}{N}}].$$

Outline of the proof of Proposition 4.2. Obtaining an upper bound on $D_{N,q}^{\text{bead}}$ is rather straightforward: we simply remove the events $\{S > I\}$ and $\{S_i \geq 0, \forall 0 \leq i \leq N + 1\}$ from the definition of $D_{N,q}^{\text{bead}}$ and then apply large deviation estimate for the signed area $A_{N+1}(S)$ of the upper envelope. The proof of the lower bound is more involved and consists of the following.

1. Getting rid of the event $\{S > I\}$ by first introducing a stronger constraint, that is, S (resp. I) remains above (resp. below) some deterministic curves.

2. Conditioning on I that remains below the deterministic curve introduced in (a) and integrating over S . Then, with the help of large deviation estimates for random walks enclosing very large area, proving that the events $\{S \in B_{N+1}^{0,+}\}$ and $\{A_{N+1}(S) = qN^2 + A_N(I)\}$ have a probability $\frac{1}{N^2} e^{-Ng(q,0)} e^{-\partial_q g(q,0) \frac{A_N(I)}{N}}$ and that the cost for S to remain above the deterministic curve introduced in (a) is bounded from below by a constant,

3. Getting rid of the additional constraint on I introduced in (a) with an FKG inequality, by observing that the random walk I whose law is penalized exponentially by $-\frac{1}{N}A_N(I)$ typically remains below its unpenalized counterpart.

Proof of the lower bound in (4.9).

Integrating on the upper envelope S . To begin with, we constrain the upper and lower envelopes to remain respectively above and below some fixed curves. In particular this allows us to handle the condition $S > I$. Define $f(t) := \gamma(t \wedge (1 - t))$, $t \in [0, 1]$, and

$$(4.11) \quad \begin{aligned} \tilde{f}_S(k) &:= (N + 1)f\left(\frac{k}{N + 1}\right) + K_S, \quad \forall 1 \leq k \leq N, \\ \tilde{f}_I(k) &:= Nf\left(\frac{k}{N}\right) + K_I, \quad \forall 1 \leq k \leq N - 1, \end{aligned}$$

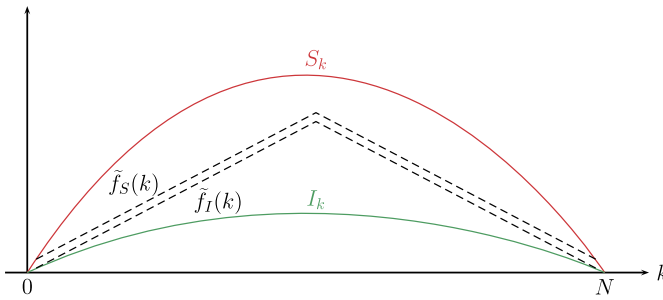


FIG. 5. Representation of the constraints on S, I , for N large. The piecewise linear curves \tilde{f}_S and \tilde{f}_I are above the wall, reaching a height of order N in the middle and bounded at both ends. S is constrained to remain above \tilde{f}_S , and I below \tilde{f}_I .

where $\gamma, K_S, K_I > 0$ are constants. If we constrain $S_k, 1 \leq k \leq N$ (resp. $I_k, 1 \leq k \leq N - 1$) to remain above $\tilde{f}_S(k)$ (resp. below $\tilde{f}_I(k)$), and provided that $K_S - K_I \geq 1 + \gamma$, then we have

$$(4.12) \quad \{S \succ I\} \supset \left\{ \begin{array}{l} S_k \geq \tilde{f}_S(k), \forall 1 \leq k \leq N; \\ I_k \leq \tilde{f}_I(k), \forall 1 \leq k \leq N - 1 \end{array} \right\}.$$

Geometrically, $\tilde{f}_\cdot(k)$ is a piecewise linear curve above the wall of order N when $k = N/2$, and constant at its ends. The constants K_S, K_I are only there for technical purposes and are mostly irrelevant when N is large (see Figure 5). More precisely, we fix γ such that Proposition 4.3 below applies (for any $K_S > 0$), then we fix K_I when applying Lemma 4.5, and we finally fix $K_S > K_I$ such that (4.12) holds.

We also add the constraint $\{A_N(I) \leq C_A N^{3/2}\}$, where $C_A > 0$ is some constant which will also be fixed when applying Lemma 4.5 below. Let us condition $D_{N,q}^{\text{bead}}$ over the trajectory I (recall (4.8)), so that we separate the constraints on I and S and we obtain the lower bound

$$(4.13) \quad D_{N,q}^{\text{bead}} \geq \mathbf{E}_{\beta,0} [e^{\delta \sum_{k=1}^N 1_{\{I_k=0\}}} \mathbf{1}_{\{I \in B_N^{0,+}\}} \mathbf{1}_{\{A_N(I) \leq C_A N^{3/2}\}} \mathbf{1}_{\{I_k \leq \tilde{f}_I(k), \forall 1 \leq k < N\}} \widehat{D}_{N,q}^{\text{bead}}(I)],$$

with:

$$(4.14) \quad \widehat{D}_{N,q}^{\text{bead}}(I) := \mathbf{E}_{\beta,0} [\mathbf{1}_{\{S_{N+1}=0\}} \mathbf{1}_{\{A_{N+1}(S) = A_N(I) + qN^2\}} \mathbf{1}_{\{S_k \geq \tilde{f}_S(k), \forall 1 \leq k < N+1\}} | I].$$

Let us fix $I = (I_k)_{k=1}^N$ which satisfies all constraints in (4.13). Proposition 4.3 below allows us to estimate $\widehat{D}_{N,q}^{\text{bead}}(I)$ up to uniform constant factors. We postpone the proof to Section 6.3.

PROPOSITION 4.3. Let $\varepsilon > 0$ and $q_1 < q_2, p_1 < p_2 \in \mathbb{R}$. There exists $\gamma > 0$ such that for all $K_S > 0$, there exist $C_2 > C_1 > 0$ and $N_0 \in \mathbb{N}$ such that for every $N \geq N_0$, one has

$$(4.15) \quad \begin{aligned} \frac{C_1}{N^2} e^{-Ng(q,p)} &\leq \mathbf{E}_{\beta,0} [\mathbf{1}_{\{X_N = pN, A_N(X) = qN^2\}} \mathbf{1}_{\{X_k \geq \tilde{f}(k), \forall 1 \leq k < N\}}] \\ &\leq \mathbf{E}_{\beta,0} [\mathbf{1}_{\{X_N = pN, A_N(X) = qN^2\}}] \leq \frac{C_2}{N^2} e^{-Ng(q,p)}, \end{aligned}$$

where we define $\tilde{f}(k) := Nf(\frac{k}{N}) + kp + K_S$, and these bounds hold uniformly in $q \in [q_1, q_2] \cap \frac{\mathbb{Z}}{2N^2}, p \in [p_1, p_2] \cap \frac{\mathbb{Z}}{N}$ satisfying $p \leq 2q - \varepsilon$.

REMARK 4.4.

(i) Although we only use Proposition 4.3 with $p = 0$, we prove it in the general case ($p \neq 0$) because such estimates will be useful when studying the model without the single-bead restriction.

(ii) The assumption $q \geq \frac{p+\varepsilon}{2}$ is required to ensure that the area enclosed in between the straight line from $(0, 0)$ to (N, pN) and a random walk trajectory from $(0, 0)$ to (N, pN) satisfying $A_N(X) = qN^2$ remains large, that is, larger than $\varepsilon N^2/2$. Such random walk trajectories are of large deviation type and concentrate on their mean profile which is a q -dependent concave curve above the straight line from $(0, 0)$ to (N, pN) and even above \tilde{f} provided γ is chosen small enough. Notice that the assumption $q = \frac{p}{2}$ would yield a trajectory with a linear mean profile from $(0, 0)$ to (N, pN) , and a lower value of q would yield a convex mean profile (below the linear trajectory).

(iii) If we remove the constraint $\{X_k \geq \tilde{f}(k), \forall 1 \leq k < N\}$ from (4.15), the proof of Proposition 4.3 was known from [7], eq. (4.14) & Theorem 4.2, for random walks on \mathbb{Z} whose increments have finite small exponential moments. In this case, the method consists in tilting the law of the N first increments in such a way that the event $(A_N, X_N) = (qN^2, pN)$ becomes typical and to apply subsequently a Gnedenko type central limit Theorem. In [4], Proposition 2.5, the case $p = 0$ with the constraint $\{X_k > 0, \forall 1 \leq k < N\}$ was considered and a lower bound $C/N^\kappa e^{-Ng(q,p)}$ was obtained with some $\kappa > 2$. Proposition 4.3 substantially reinforces this latter lower bound by proving that the constraint $\{X_k \geq \tilde{f}(k), \forall 1 \leq k < N\}$ only costs up to a constant factor. Heuristically, this comes from the fact that after tilting the increments of the random walk in such a way that their expected profile matches the prescription $(A_N, X_N) = (qN^2, pN)$, the increments close to the origin (resp. close to the endpoint) have a positive drift uniformly bounded from below by a positive constant (resp. a negative drift bounded from above by a negative constant) and therefore the walk does not return to the origin between time 1 and $N - 1$ with positive probability.

Recall (4.14), and let $p' = 0, q' := (qN^2 + A_N(I))/(N + 1)^2$. Under our assumptions, there is a compact subset $[q'_1, q'_2] \subset (0, \infty)$ and $N_0 \in \mathbb{N}$ such that $q' \in [q'_1, q'_2]$ for all $N \geq N_0, q \in [q_1, q_2]$ and I satisfying all constraints from (4.13) (recall that $A_N(I) = O(N^{3/2})$). Moreover, we have $0 = p' < q'_1/2$, hence we can apply Proposition 4.3 to $\widehat{D}_{N,q}^{\text{bead}}(I)$, and we obtain the uniform bound for $N \geq N_0$,

$$(4.16) \quad \widehat{D}_{N,q}^{\text{bead}}(I) \geq \frac{C_3}{N^2} e^{-Ng(q',0)}.$$

Now recall that g is C^2 and convex, so there exists $C_3 > 0$ such that

$$g(q', 0) \leq g(q, 0) + \partial_q g(q, 0)(q' - q) + C_3(q' - q)^2,$$

uniformly in $q \in [q_1, q_2]$ and $q' \in [q'_1, q'_2]$. Recall that $A_N(I) = O(N^{3/2})$, and notice $q' - q = \frac{A_N(I)}{N^2} + O(1/N)$. Thereby (4.16) becomes

$$(4.17) \quad \widehat{D}_{N,q}^{\text{bead}}(I) \geq \frac{C_4}{N^2} e^{-Ng(q,0) - \partial_q g(q,0) \frac{A_N(I)}{N}},$$

where C_4 is uniform in $q \in [q_1, q_2] \cap \frac{\mathbb{N}}{2N^2}$ and I . Plugging (4.17) into (4.13), we obtain

$$(4.18) \quad \begin{aligned} D_{N,q}^{\text{bead}} &\geq \frac{C_4}{N^2} e^{-Ng(q,0)} \mathbf{E}_{\beta,0} [e^{\delta \sum_{k=1}^N \mathbf{1}_{\{I_k=0\}}} \mathbf{1}_{\{I \in B_N^{0,+}\}} \mathbf{1}_{\{A_N(I) \leq C_A N^{3/2}\}} \\ &\quad \times \mathbf{1}_{\{I_k \leq \tilde{f}_I(k), \forall 1 \leq k < N\}} e^{-\partial_q g(q,0) \frac{A_N(I)}{N}}]. \end{aligned}$$

Recall that g is nonnegative, and let us point out that $\partial_q g(q, 0) = \tilde{h}_0^{q,0} > 0$ for all $q > 0$ (recall (3.12), and see [4] or Section 6.3, Rem. 5.5).

Relaxing some constraints on the lower envelope I. Our goal now is to drop the events $\{I_k \leq \tilde{f}_I(k), \forall 1 \leq k < N\}$ and $\{A_N(I) \leq C_A N^{3/2}\}$ in the right-hand side in (4.18) by paying up to a constant factor uniform in $N \geq N_0$ and $q \in [q_1, q_2]$. To that aim, we use an FKG inequality. Indeed, notice that the functions

$$x = (x_k)_{k=1}^N \longmapsto e^{\delta \sum_{k=1}^N \mathbf{1}_{\{x_k=0\}}} e^{-\partial_q g(q,0) \frac{A_N(x)}{N}},$$

$$\text{and } x \longmapsto \mathbf{1}_{\{x_k \leq \tilde{f}_I(k), \forall 1 \leq k < N\}} \mathbf{1}_{\{A_N(x) \leq C_A N^{3/2}\}},$$

are both bounded and *nonincreasing* on $B_N^{0,+}$, where we say that $f : B_N^{0,+} \rightarrow \mathbb{R}$ is nonincreasing if for all $x, y \in B_N^{0,+}$ such that $x_k \leq y_k, \forall 1 \leq k \leq N$, one has $f(x) \geq f(y)$. Thereby the FKG inequality claimed in Proposition B.1 yields that

$$(4.19) \quad \mathbf{E}_{\beta,0} \left[e^{\delta \sum_{k=1}^N \mathbf{1}_{\{x_k=0\}}} \mathbf{1}_{\{x_k \leq \tilde{f}_I(k), \forall 1 \leq k < N; A_N(x) \leq C_A N^{3/2}\}} e^{-\partial_q g(q,0) \frac{A_N(x)}{N}} \mid X \in B_N^{0,+} \right]$$

$$\geq \mathbf{P}_{\beta,0}(X_k \leq \tilde{f}_I(k), \forall 1 \leq k < N; A_N(X) \leq C_A N^{3/2} \mid X \in B_N^{0,+})$$

$$\times \mathbf{E}_{\beta,0} \left[e^{\delta \sum_{k=1}^N \mathbf{1}_{\{x_k=0\}}} e^{-\partial_q g(q,0) \frac{A_N(x)}{N}} \mid X \in B_N^{0,+} \right].$$

Finally, the first factor of the right-hand side is bounded from below by some constant (close to 1) by the following lemma, which is proven in Section 6.1.

LEMMA 4.5. *Let $\varepsilon, \gamma > 0$. There exist $C_A, K_I > 0$ and $N_0 \in \mathbb{N}$ such that*

$$(4.20) \quad \mathbf{P}_{\beta,0}(X_k \leq \tilde{f}_I(k), \forall 1 \leq k < N; A_N(X) \leq C_A N^{3/2} \mid X \in B_N^{0,+}) \geq 1 - \varepsilon,$$

for all $N \geq N_0$.

By recollecting (4.18) and (4.19), and provided that C_A, K_I are sufficiently large to apply Lemma 4.5, we conclude that for every $N \geq N_0$ and every $q \in [q_1, q_2] \cap \frac{\mathbb{N}}{2N^2}$ we have

$$(4.21) \quad D_{N,q}^{\text{bead}} \geq \frac{C_5}{N^2} e^{-Ng(q,0)} \mathbf{E}_{\beta,0} \left[e^{\delta \sum_{k=1}^N \mathbf{1}_{\{I_k=0\}}} \mathbf{1}_{\{I \in B_N^{0,+}\}} e^{-\partial_q g(q,0) \frac{A_N(I)}{N}} \right],$$

which completes the proof of the lower bound. \square

Proof of the upper bound in (4.9). We recall (4.8) and we bound $D_{N,q}^{\text{bead}}$ from above by relaxing partially the constraints on S -more precisely the constraints $\{S > I\}$ and $\{S_i \geq 0, \forall 1 \leq i \leq N\}$. Therefore,

$$(4.22) \quad D_{N,q}^{\text{bead}} \leq \mathbf{E}_{\beta,0} \left[e^{\delta \sum_{k=1}^N \mathbf{1}_{\{I_k=0\}}} \mathbf{1}_{\{I \in B_N^{0,+}\}} \mathbf{1}_{\{S_{N+1}=0\}} \mathbf{1}_{\{A_{N+1}(S) = A_N(I) + qN^2\}} \right],$$

$$= \mathbf{E}_{\beta,0} \left[e^{\delta \sum_{k=1}^N \mathbf{1}_{\{I_k=0\}}} \mathbf{1}_{\{I \in B_N^{0,+}\}} \tilde{D}_{N,q}^{\text{bead}}(I) \right].$$

with

$$\tilde{D}_{N,q}^{\text{bead}}(I) := \mathbf{P}_{\beta,0}[S_{N+1} = 0, A_{N+1}(S) = A_N(I) + qN^2 \mid I].$$

In order to get uniform bounds on $\tilde{D}_{N,q}^{\text{bead}}(I)$ with Proposition 4.3, we need to drop those I sweeping a too large area. To that aim, for any $c > 0$, we rewrite (4.22) as $\tilde{D}_{N,q}^{\text{bead}} \leq R_{N,q}^1(c) + R_{N,q}^2(c)$ with

$$R_{N,q}^1(c) := \mathbf{E}_{\beta,0} \left[e^{\delta \sum_{k=1}^N \mathbf{1}_{\{I_k=0\}}} \mathbf{1}_{\{I \in B_N^{0,+}, A_N(I) \leq cN^2\}} \tilde{D}_{N,q}^{\text{bead}}(I) \right],$$

$$R_{N,q}^2(c) := \mathbf{E}_{\beta,0} \left[e^{\delta \sum_{k=1}^N \mathbf{1}_{\{I_k=0\}}} \mathbf{1}_{\{I \in B_N^{0,+}, A_N(I) > cN^2\}} \tilde{D}_{N,q}^{\text{bead}}(I) \right],$$

and we write the very crude bound

$$\begin{aligned}
 R_{N,q}^2(c) &\leq e^{\delta N} \mathbf{P}_{\beta,0}[A_N(I) > cN^2] \leq e^{\delta N} \mathbf{P}_{\beta,0}\left[\max_{1 \leq i \leq N} I_i > cN\right] \\
 (4.23) \qquad &\leq e^{\delta N} \mathbf{P}_{\beta,0}\left[\sum_{i=1}^N |I_i - I_{i-1}| > cN\right].
 \end{aligned}$$

Then, we set $M := 1 + \max\{g(q, 0), q \in [q_1, q_2]\}$ and we use the fact that $(I_i - I_{i-1})_{i \geq 1}$ are i.i.d. with finite small exponential moments to conclude (via a Markov exponential inequality) that there exists a $c_{M,\delta} > 0$ such that for every $N \geq N_0$ and $q \in [q_1, q_2] \cap \frac{\mathbb{N}}{2N^2}$,

$$(4.24) \qquad R_{N,q}^2(c_{M,\delta}) \leq e^{-MN}.$$

Let us now consider $R_{N,q}^1(c_{M,\delta})$ and use Proposition 4.3 to assert that there exists $C_6 > 0$ and $N_0 \in \mathbb{N}$ (depending on M) such that for every $N \geq N_0, q \in [q_1, q_2] \cap \frac{\mathbb{N}}{2N^2}$ and I satisfying $A_N(I) \leq c_{M,\delta}N^2$ we have

$$\tilde{D}_{N,q}^{\text{bead}}(I) \leq \frac{C_6}{N^2} e^{-Ng(q',0)},$$

with $q' = (qN^2 + A_N(I))/(N + 1)^2$. Recall that g is convex, therefore we can bound $g(q', 0)$ from below with

$$g(q', 0) \geq g(q, 0) + \partial_q g(q, 0)(q' - q).$$

Moreover $q' - q = \frac{A_N(I)}{N^2} + O(1/N)$, thus,

$$\begin{aligned}
 R_{N,q}^1(c_{M,\delta}) &\leq \frac{C_7}{N^2} e^{-g(q,0)N} \\
 (4.25) \qquad &\times \mathbf{E}_{\beta,0}\left[e^{\delta \sum_{k=1}^N \mathbf{1}_{\{I_k=0\}}} \mathbf{1}_{\{I \in B_N^{0,+}, A_N(I) \leq c_{M,\delta}N^2\}} e^{-\partial_q g(q,0) \frac{A_N(I)}{N}}\right] \\
 &\leq \frac{C_7}{N^2} e^{-g(q,0)N} E_{N,q}^{\text{bead}},
 \end{aligned}$$

where we recall (4.10).

Our proof will be complete once we show that for N large enough and for every $q \in [q_1, q_2] \cap \frac{\mathbb{N}}{2N^2}$, the right-hand side in (4.24) (i.e., e^{-MN}) is not larger than the right-hand side in (4.25). To that aim, we write the lower bound

$$\begin{aligned}
 (4.26) \qquad E_{N,q}^{\text{bead}} &\geq \mathbf{P}_{\beta,0}(A_N(I) \leq C_A N^{3/2} \mid I \in B_N^{0,+}) \\
 &\times \mathbf{P}_{\beta,0}(I \in B_N^{0,+}) e^{-\max\{\partial_q g(q,0), q \in [q_1, q_2]\} C_A \sqrt{N}},
 \end{aligned}$$

for any $C_A > 0$, where we constrained the walk to have area at most $C_A N^{3/2}$. A direct consequence of Lemma 4.5 is that there exist $C_A, C_8 > 0$ such that for N large enough,

$$\mathbf{P}_{\beta,0}(A_N(I) \leq C_A N^{3/2} \mid I \in B_N^{0,+}) \geq C_8,$$

(more generally this follows from an invariance principle on nonnegative random walk bridges, see (6.2) below). The second factor in (4.26) is bounded from below by $C_9 N^{-3/2}$ for some C_9 (see (6.1)). Recalling that $q \mapsto \partial_q g(q, 0)$ is continuous on $[a_1, a_2]$, we conclude that there exists $C_{10}, c_1 > 0$ such that for N large enough and every $q \in [q_1, q_2] \cap \frac{\mathbb{N}}{2N^2}$, we have

$$(4.27) \qquad E_{N,q}^{\text{bead}} \geq \frac{C_{10}}{N^{3/2}} e^{-c_1 \sqrt{N}}.$$

Recalling that $M = 1 + \max\{g(q, 0), q \in [q_1, q_2]\} > 0$, it suffices to combine (4.24) with (4.25) and (4.27) to complete the proof of the upper bound in (4.9). \square

4.3. *Step 3: Area-penalized wetting model for $\delta > \tilde{\delta}_c(\beta)$.* In this section, we give estimates on $E_{N,q}^{\text{bead}}$ when $\delta > \tilde{\delta}_c(\beta)$ —in particular in the phase \mathcal{AC} . Notice that $E_{N,q}^{\text{bead}}$ is the partition function of a wetting model with an additional pre-wetting term. Let $Z_{\text{wet},N}^{\beta,\delta}$ be the standard wetting partition function (without pre-wetting, see (A.1)). Such wetting models have already been studied extensively by mathematicians—we provide well-known results on them in Proposition A.1. Recall that $\tilde{\delta}_c(\beta) = \inf\{\delta > 0, h_\beta(\delta) > 0\}$ is the critical point of the wetting model $Z_{\text{wet},N}^{\beta,\delta}$, $h_\beta(\delta)$ is its free energy, and \asymp is defined in (0.2).

PROPOSITION 4.6. *Let $q_2 > q_1 > 0$, and assume $\delta > \tilde{\delta}_c(\beta)$. Then $E_{N,q}^{\text{bead}} \asymp e^{h_\beta(\delta)N}$ uniformly in $q \in [q_1, q_2]$.*

PROOF. The upper bound is a straightforward consequence of the inequality $E_{N,q}^{\text{bead}} \leq Z_{\text{wet},N}^{\beta,\delta}$ and Proposition A.1 (ii). For the lower bound, we first bound $\partial_q g(q, 0)$ by some $c > 0$ for all $q \in [q_1, q_2]$ (recall that it is a continuous and positive function), then decompose trajectories in $E_{N,q}^{\text{bead}}$ into excursions:

$$(4.28) \quad E_{N,q}^{\text{bead}} \geq \sum_{r=1}^N \sum_{\substack{1 \leq t_1, \dots, t_r \leq N, \\ t_1 + \dots + t_r = N}} \prod_{i=1}^r K_\beta(t_i) e^\delta \mathbf{E}_{\beta,0} \left[e^{-c \frac{A_{t_i}(I)}{N}} \mid \tau = t_i, I_{t_i} = 0 \right],$$

where we define $\tau = \inf\{t \geq 1; I_t \leq 0\}$ and $K_\beta(t) = \mathbf{P}_{\beta,0}(\tau = t, I_t = 0)$, $t \geq 1$ as in Appendix A. Using Jensen’s inequality, we can claim that

$$(4.29) \quad \mathbf{E}_{\beta,0} \left[e^{-c \frac{A_t(I)}{N}} \mid \tau = t, I_t = 0 \right] \geq e^{-c \frac{1}{N} \mathbf{E}_{\beta,0}[A_t(I) \mid \tau = t, I_t = 0]}, \quad t \in \mathbb{N}.$$

Moreover, by using the inequality $\mathbf{E}_{\beta,0}(|A_t(I)|) \leq \mathbf{E}_{\beta,0}(|I_1|) \sum_{j=1}^t j$ combined with (A.2) we can state that there exists a $\bar{c} > 0$ such that

$$(4.30) \quad \mathbf{E}_{\beta,0} [A_t(I) \mid \tau = t, I_t = 0] \leq \mathbf{E}_{\beta,0} (|A_t(I)|) K_\beta(t)^{-1} \leq \bar{c} t^{7/2}, \quad t \in \mathbb{N}.$$

Using (4.28)–(4.30), we obtain that there exists a $c_1 > 0$ such that

$$(4.31) \quad E_{N,q}^{\text{bead}} \geq \sum_{r=1}^N \sum_{\substack{1 \leq t_1, \dots, t_r \leq N, \\ t_1 + \dots + t_r = N}} \prod_{i=1}^r K_\beta(t_i) e^\delta e^{-c_1 \frac{t_i^{7/2}}{N}}.$$

Since $\delta > \tilde{\delta}_c(\beta)$, equation (A.3) guarantees that

$$(4.32) \quad Q_\infty(t) := K_\beta(t) e^\delta e^{-h_\beta(\delta)t}, \quad t \in \mathbb{N},$$

is a probability distribution on \mathbb{N} . We also define, for all $N \in \mathbb{N}$,

$$(4.33) \quad Q_N(t) := K_\beta(t) e^\delta e^{-h_N t} e^{-\frac{c_1}{N} t^{7/2}}, \quad t \in \mathbb{N},$$

where h_N is the unique solution of

$$(4.34) \quad \sum_{t \geq 1} K_\beta(t) e^\delta e^{-h_N t} e^{-\frac{c_1}{N} t^{7/2}} = 1.$$

Notice that $h_N \leq h_\beta(\delta)$ and $h_N \rightarrow h_\beta(\delta)$ as $N \rightarrow \infty$. Plugging these notations into (4.31), we have

$$(4.35) \quad E_{N,q}^{\text{bead}} \geq e^{h_N N} Q_N \quad (N \in \tilde{\tau}),$$

where $\tilde{\tau}$ is a renewal process in \mathbb{N} whose inter-arrival distribution is given by Q_N .

LEMMA 4.7. *There is $C_{11} > 0$ such that, for all $N \in \mathbb{N}$,*

$$(4.36) \quad 0 \leq h_\beta(\delta) - h_N \leq \frac{C_{11}}{N}.$$

PROOF. We already stated that $h_N \leq h_\beta(\delta)$ for all $N \in \mathbb{N}$. Using that Q_∞ and Q_N are probability distributions on \mathbb{N} , we write

$$\sum_{t \geq 1} K_\beta(t) e^\delta e^{-h_\beta(\delta)t} (e^{(h_\beta(\delta)-h_N)t} - 1) = \sum_{t \geq 1} K_\beta(t) e^\delta e^{-h_N t} (1 - e^{-\frac{c_1}{N} t^{7/2}}) \geq 0.$$

We bound the left-hand side from below using $e^{(h_\beta(\delta)-h_N)t} - 1 \geq (h_\beta(\delta) - h_N)t \geq h_\beta(\delta) - h_N$ for all $t \geq 1$ and (4.32), so

$$h_\beta(\delta) - h_N \leq \sum_{t \geq 1} K_\beta(t) e^\delta e^{-h_N t} (1 - e^{-\frac{c_1}{N} t^{7/2}}).$$

Since $\lim_{N \rightarrow \infty} h_N = h_\beta(\delta)$, we have some $N_0 \in \mathbb{N}$ such that $h_N \geq h_\beta(\delta)/2$ for all $N \geq N_0$. Moreover, $1 - e^{-\frac{c_1}{N} t^{7/2}} \leq \frac{c_1}{N} t^{7/2}$, so for all $N \geq N_0$,

$$h_\beta(\delta) - h_N \leq \frac{c_1}{N} \sum_{t \geq 1} K_\beta(t) e^\delta e^{-(h_\beta(\delta)/2)t} t^{7/2},$$

and that sum is finite, which concludes the proof. \square

Applying this lemma to (4.35), we obtain

$$(4.37) \quad E_{N,q}^{\text{bead}} \geq C_{12} e^{h_\beta(\delta)N} Q_N \quad (N \in \tau),$$

and therefore the proof will be complete once we show that $\liminf_N Q_N(N \in \tau) > 0$. Applying the main theorem from [22], there is M a random variable with distribution $\text{Geom}(b)$, $b \in (0, 1)$, and a sequence $Z_i = C' + \zeta_i$, $i \geq 1$ where $C' > 0$ and $(\zeta_i)_{i \in \mathbb{N}}$ are i.i.d. variables with distribution $\text{Exp}(a)$, $a > 0$, such that M and $(Z_i)_{i \in \mathbb{N}}$ are independent, and

$$(4.38) \quad \left| Q_N(n \in \tau) - \frac{1}{\mu_N} \right| \leq C_{13} \mathbf{P} \left(\sum_{i=1}^M Z_i > n \right),$$

for all $n \in \mathbb{N}$ and $N \in \mathbb{N}$, where $\mu_N := \mathbf{E}_{Q_N}[\tau_1] \in (0, \infty)$. The key feature here is that $C', C_{13} > 0, a > 0$ and $b \in (0, 1)$ can be taken uniformly in $N \in \mathbb{N}$. We do not write the details here as it suffices to check the proof in [22]—more precisely one has to use that $Q_N(t) \leq C_{14} Q_\infty(t)$ uniformly in $N, t \in \mathbb{N}$ (this follows from Lemma 4.7) in [22], (2.9), and $Q_N(t) \geq C_{15} Q_\infty(t)$ uniformly in $N \in \mathbb{N}$ and finitely many $t \in \mathbb{N}$ in [22], (2.13).

In particular when $n = N$, the right-hand side in (4.38) decays to 0 as $N \rightarrow \infty$, so $Q_N(N \in \tau) - \frac{1}{\mu_N}$ converges to 0 as $N \rightarrow \infty$. Finally, we note that $\lim_{N \rightarrow \infty} \mu_N = \mu_\infty := \mathbf{E}_{Q_\infty}[\tau_1] \in (0, \infty)$ (by dominated convergence theorem). Thus, $Q_N(N \in \tau) \geq \frac{1}{2\mu_\infty}$ for N sufficiently large, and this concludes the proof of Proposition 4.6. \square

4.4. *Step 4: Area-penalized wetting model for $\delta \leq \tilde{\delta}_c(\beta)$.* Estimates of $E_{N,q}^{\text{bead}}$ in the phase \mathcal{DC} are more involved than in \mathcal{AC} . Actually we only manage to find a sharp asymptotic of $E_{N,q}^{\text{bead}}$ when $\delta = 0$. To lighten upcoming notations, let us define for all $\gamma > 0$,

$$(4.39) \quad E_N(\gamma) := \mathbf{E}_{\beta,0} [e^{-\gamma \frac{A_N(t)}{N}} \mathbf{1}_{\{I \in B_N^{0,+}\}}].$$

Recall that σ_β^2 is the variance of I_1 under \mathbf{P}_β .

For any $T > 0$ and $y \geq 0$, let $(\mathcal{M}_{s,T}^y)_{s \in [0,T]}$ denote the Brownian meander on $[0, T]$ starting from y —henceforth we denote its law with \mathbb{P} and its expectation with \mathbb{E} . When $y > 0$, it has the same law as the Brownian motion starting from y and conditioned to remain positive on $[0, T]$. Define, for all $\gamma > 0$,

$$(4.40) \quad J(\gamma) := \lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}[e^{-\gamma \int_0^T \mathcal{M}_{s,T}^0 ds}] = -2^{-1/3} |a_1| \gamma^{2/3},$$

where a_1 denotes the smallest zero (in absolute value) of the Airy function. More precisely, one notices $\mathbb{E}[e^{-\gamma \int_0^T \mathcal{M}_{s,T}^0 ds}] = \mathbb{E}[e^{-\gamma T^{3/2} \int_0^1 \mathcal{M}_{s,1}^0 ds}]$, and the latter expectation has been computed analytically in [29] (see also [17], (209)), which leads to the second identity. We claim the following.

PROPOSITION 4.8. *One has*

$$(4.41) \quad \frac{1}{N^{1/3}} \log E_N(\gamma) \xrightarrow{N \rightarrow \infty} J(\sigma\beta\gamma),$$

locally uniformly in $\gamma \in (0, \infty)$.

REMARK 4.9. In [15], Lemma 1 and Theorem 2 provide a result that is close in spirit to our Proposition 4.8 but in a broader framework. To be more specific, the authors consider a random walk $X = (X_i)_{i=0}^N$ on \mathbb{Z} with i.i.d. increments that are symmetric with finite exponential moments. Let us denote by \mathbb{p} the law of the random walk and by $\mathbf{E}_{\mathbb{p}}$ its associated expectation. For $\lambda > 0$ the following partition function is considered:

$$(4.42) \quad Z_{N,+,\lambda}(\mathbb{X}) = \mathbf{E}_{\mathbb{p}}[e^{-\sum_{i=1}^N V_\lambda(X_i)} \mathbf{1}_{\{X \in B_N^{0,+}\}}],$$

where $V_\lambda : [0, \infty) \mapsto [0, \infty)$ is a continuous increasing function satisfying $V_\lambda(0) = 0$ and $\lim_{x \rightarrow \infty} V_\lambda(x) = \infty$. The free energy associated with the partition function in (4.42) is denoted by e_λ . With [15], Lemma 1, the authors prove that e_λ is bounded away from 0 and ∞ as $\lambda \searrow 0$ and subsequently, with Theorem 2, they identify $\lim_{\lambda \searrow 0} e_\lambda$ by using tools from spectral theory. In that latter framework, our Proposition 4.8 would correspond to $V_\lambda(x) = \lambda x$ in the case where λ decays with the system size, that is, $\lambda = \gamma/N$.

The proof of Lemma 1 in [15] is very different from that of Proposition 4.8, the common point though being the decomposition of $\{0, \dots, N\}$ into blocks whose size are comparable to the correlation length of the system ($N^{2/3}$ here and H_λ^2 the solution in H of $H^2 V_\lambda(H) = 1$ in [15]). On such blocks, an invariance principle can be applied to the random walk rescaled in time/space by $N^{2/3}$ and $N^{1/3}$ here and by H_λ^2 and H_λ in [15].

PROOF. First, we claim that it suffices to prove the pointwise convergence. Indeed, one notices that $\gamma \mapsto J(\sigma\beta\gamma)$ is continuous, and $\gamma \mapsto \frac{1}{N^{1/3}} \log E_N(\gamma)$ is nonincreasing for any $N \in \mathbb{N}$. It is well known that those assumptions put together with the pointwise convergence imply the locally uniform convergence; see, for example, [27], Proposition 2.1, for a proof.

Notice also that $\frac{1}{\sigma\beta} I$ is a random walk on $\frac{1}{\sigma\beta} \mathbb{Z}$ with variance 1, and upcoming computations still hold when replacing γ with $\sigma\beta\gamma$. Hence, we can assume without loss generality that $\sigma\beta = 1$.

Upper bound. Let $N \in \mathbb{N}$ and $T > 0$, and let us denote $N_T := \lfloor TN^{2/3} \rfloor$. Set also $a_N := \lfloor N/N_T \rfloor$. We decompose a trajectory contributing to $E_N(\gamma)$ into a_N blocks of length N_T and a remaining block of length at most N_T . We apply Markov property at times jN_T for $j \in \{1, \dots, a_N\}$ and we bound the contribution of the very last block by 1 to obtain

$$(4.43) \quad E_N(\gamma) \leq \left(\sup_{x \in \mathbb{N}} \mathbf{E}_{\beta,x} [e^{-\frac{\gamma}{N} A_{N_T}(I)} \mathbf{1}_{\{I \in B_{N_T}^+\}}] \right)^{a_N},$$

where for $n \in \mathbb{N}$, $B_n^+ := \{(X_k)_{k=1}^n \in \mathbb{Z}^n; X_k \geq 0 \forall 1 \leq k \leq n\}$ and where, for $x \in \mathbb{Z}$, we recall the definitions of $\mathbf{P}_{\beta,x}$ and $\mathbf{E}_{\beta,x}$ from (2.8).

LEMMA 4.10. For $\alpha > 0$ and $m \in \mathbb{N}_0$,

$$x \mapsto \mathbf{E}_{\beta,x}[e^{-\alpha A_m(I)} | I \in B_m^+],$$

is nonincreasing on \mathbb{N}_0 . This also holds when conditioning over $B_m^{0,+}$ instead of B_m^+ .

We postpone the proof of this lemma to Section 6.2. Choosing $\alpha = \gamma/N$ and applying $\frac{1}{N^{1/3}} \log$ to (4.43) (and because $\mathbf{P}_{\beta,x}(B_m^+) \leq 1$), we obtain

$$\begin{aligned} \frac{1}{N^{1/3}} \log E_N(\gamma) &\leq \frac{a_N}{N^{1/3}} \log \left(\sup_{x \in \mathbb{N}} \mathbf{E}_{\beta,x} [e^{-\frac{\gamma}{N} A_{N_T}(I)} | I \in B_{N_T}^+] \right) \\ (4.44) \qquad \qquad \qquad &\leq \left(1 + \frac{1}{N_T} \right) \frac{1}{T} \log \mathbf{E}_{\beta,0} [e^{-\frac{\gamma}{N} A_{N_T}(I)} | I \in B_{N_T}^+], \end{aligned}$$

for all N, T . Moreover the process $(\frac{1}{N^{1/3}} I_{\lfloor sN^{2/3} \rfloor})_{s \in [0, T]}$ conditioned to remain nonnegative converges weakly as $N \rightarrow \infty$ to the Brownian meander $(\mathcal{M}_{s,T}^0)_{s \in [0, T]}$ in the set of cadlag functions on $[0, T]$ endowed with the uniform convergence topology (see [2], Theorem 3.2), and the swiped area is a continuous function of the trajectory. Thereby,

$$(4.45) \qquad \lim_{N \rightarrow \infty} \mathbf{E}_{\beta,0} [e^{-\frac{\gamma}{N} A_{N_T}(I)} | I \in B_{N_T}^+] = \mathbb{E} \left[\exp \left(-\gamma \int_0^T \mathcal{M}_{s,T}^0 ds \right) \right],$$

for all $\gamma > 0$. Recollecting (4.44), we have for any fixed $T > 0$,

$$\limsup_{N \rightarrow \infty} \frac{1}{N^{1/3}} \log E_N(\gamma) \leq \frac{1}{T} \log \left(\mathbb{E} \left[\exp \left(-\gamma \int_0^T \mathcal{M}_{s,T}^0 ds \right) \right] \right).$$

Choosing $T > 0$ arbitrarily large, we conclude

$$\limsup_{N \rightarrow \infty} \frac{1}{N^{1/3}} \log E_N(\gamma) \leq J(\gamma).$$

Lower bound. It remains to prove that

$$(4.46) \qquad \liminf_{N \rightarrow \infty} \frac{1}{N^{1/3}} \log E_N(\gamma) \geq J(\gamma).$$

Let us settle some notation, that is, for $\kappa_1 < \kappa_2 \in \mathbb{R}$ and $N \in \mathbb{N}$ we define $\mathcal{O}_{\kappa_1, \kappa_2, N} := [\kappa_1 N^{\frac{1}{3}}, \kappa_2 N^{\frac{1}{3}}]$. For $\Delta > 0$, $N, n \in \mathbb{N}$ and $x, y \in \mathbb{N}_0$ we set

$$(4.47) \qquad \begin{aligned} G_{N,n}^{x,y} &:= \mathbf{E}_{\beta,x} [e^{-\frac{\gamma}{N} A_n(I)} \mathbf{1}_{\{I \in B_n^{y,+}\}}], \\ \text{and } \tilde{G}_{N,\Delta,n}^x &:= \sum_{y \in \mathcal{O}_{\Delta/2, \Delta, N}} G_{N,n}^{x,y}. \end{aligned}$$

Let also $N \in \mathbb{N}$ and $T > 0$, and define $N_T := \lfloor TN^{2/3} \rfloor$ and $a_N := \lfloor (N - 2N_T)/(N_T + 2) \rfloor$ (where $a_N \geq 0$ as soon as $N \geq 8T^3$). We decompose a trajectory in $E_N(\gamma)$ into $a_N + 2$ blocks: the first block has length N_T , the a_N following blocks have length $N_T + 2$, and the final block has length ρN_T for some $\rho \in [1, 2 + \frac{1}{N_T}]$. We restrict $E_N(\gamma)$ to those trajectories located inside $\mathcal{O}_{\Delta/2, \Delta, N}$ at times N_T and $(j + 1)N_T + 2j$ for every $j \in \{1, \dots, a_N\}$. Then, applying Markov property at those times we obtain

$$(4.48) \qquad E_N(\gamma) \geq \tilde{G}_{N,\Delta,N_T}^0 [\tilde{G}_{N,\Delta,N_T+2}^{a_N}] \inf_{x \in \mathcal{O}_{\Delta/2, \Delta, N}} G_{N,\rho N_T}^{x,0},$$

with

$$(4.49) \quad \widehat{G}_{N,\Delta,n} := \inf_{x \in \mathcal{O}_{\Delta/2,\Delta,N}} \widetilde{G}_{N,\Delta,n}^x.$$

We prove (4.46) subject to claims 1, 2, 3 and 4 below. Before stating those claims, we need some additional notation. For $\Delta > 0$, $n \in \mathbb{N}$ and $N \in \mathbb{N}$, we set

$$(4.50) \quad U_{N,\Delta,n}^j := \mathbf{E}_{\beta,0} [e^{-\frac{\gamma}{N} A_n(I)} \mathbf{1}_{\{I \in \widetilde{B}_{n,\Delta N^{1/3}}^+\}} \mathbf{1}_{\{I_n \in \widetilde{\mathcal{O}}_{\Delta,N}^j\}}], \quad j \in \{1, 2, 3\},$$

with

$$\begin{aligned} \widetilde{\mathcal{O}}_{\Delta,N}^1 &:= \left(\frac{-\Delta N^{1/3}}{2}, \frac{-\Delta N^{1/3}}{4} \right), \\ \widetilde{\mathcal{O}}_{\Delta,N}^2 &:= \left[\frac{-\Delta N^{1/3}}{4}, 0 \right), \quad \text{and} \quad \widetilde{\mathcal{O}}_{\Delta,N}^3 := \left[0, \frac{\Delta N^{1/3}}{4} \right], \end{aligned}$$

and where we introduce for $\kappa \geq 0$ and $n \in \mathbb{N}$,

$$\widetilde{B}_{n,\kappa}^+ := \left\{ (X_i)_{i=1}^n \in \mathbb{Z}^n; X_i \geq -\frac{\kappa}{2}, \forall 1 \leq i < n, X_n > -\frac{\kappa}{2} \right\}.$$

Claim 1 handles the first and last factors in (4.48). Claim 2 combined with Claim 3 allows us to get rid of the infimum in the definition of $\widehat{G}_{N,\Delta,n}$ in (4.49) and replace it by $U_{N,\Delta,n}^1 + U_{N,\Delta,n}^2$. On this latter quantity one may apply an FKG inequality and get rid of the constraint $\{I_n \in \mathcal{O}_{\Delta/2,\Delta,N}\}$ in $\widehat{G}_{N,\Delta,n}$. Claim 4 is used at the end of the proof to retrieve $J(\gamma)$.

CLAIM 1. For $\Delta > 0$ and $T > 0$, one has

$$(4.51) \quad \lim_{N \rightarrow \infty} \frac{1}{N^{1/3}} \log \widetilde{G}_{N,\Delta,\rho N_T}^0 = \lim_{N \rightarrow \infty} \frac{1}{N^{1/3}} \log \left(\inf_{x \in \mathcal{O}_{\Delta/2,\Delta,N}} G_{N,\rho N_T}^{x,0} \right) = 0,$$

uniformly in $\rho \in [1, 3]$.

CLAIM 2. For $\Delta > 0$ and $N, n \in \mathbb{N}$,

$$(4.52) \quad \widehat{G}_{N,\Delta,n} \geq \exp\left(-\frac{\gamma \Delta n}{N^{2/3}}\right) \min\{U_{N,\Delta,n}^2, U_{N,\Delta,n}^3\}.$$

CLAIM 3. There exists a $C > 0$ such that for $\Delta > 0$ and $N, n \in \mathbb{N}$,

$$(4.53) \quad U_{N,\Delta,n}^j \leq C \exp\left(\frac{\gamma \Delta n}{N^{2/3}}\right) U_{N,\Delta,n+1}^{j+1}, \quad \text{for } j \in \{1, 2\}.$$

CLAIM 4. One has

$$(4.54) \quad \limsup_{\Delta \rightarrow 0^+} \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}[e^{-\gamma \int_0^T \mathcal{M}_{s,T}^\Delta ds}] \geq J(\gamma).$$

We resume the proof of the lower bound. We observe that by constraining the last increment of the random walk I to be null in $U_{N,\Delta,n+1}^2$ we get the inequality

$$(4.55) \quad U_{N,\Delta,n}^2 \leq c_\beta U_{N,\Delta,n+1}^2.$$

Then, using Claim 3 for $n = N_T$ and $j = 1$ and using (4.55) for $n = T N^{2/3}$, we obtain that there exists a $C_1 > 0$ such that for $T, \Delta > 0$ and $N \in \mathbb{N}$,

$$(4.56) \quad U_{N,\Delta,N_T}^1 + U_{N,\Delta,N_T}^2 \leq C_1 e^{\gamma \Delta T} U_{N,\Delta,N_T+1}^2.$$

Then, it suffices to apply, on the one hand, Claim 3 for $j = 2$ and $n = N_T + 1$ to the right-hand side in (4.56) and, on the other hand, (4.55) for $n = N_T + 1$ to the right-hand side in (4.56) to assert that there exists a $C_2 > 0$ such that for $T, \Delta > 0$ and $N \in \mathbb{N}$,

$$(4.57) \quad V_{N_T} := U_{N,\Delta,N_T}^1 + U_{N,\Delta,N_T}^2 \leq C_2 e^{2\gamma\Delta T} \min\{U_{N,\Delta,N_T+2}^2, U_{N,\Delta,N_T+2}^3\}.$$

We note that V_{N_T} may be rewritten under the form

$$(4.58) \quad \begin{aligned} V_{N_T} &= \mathbf{E}_{\beta,0}[e^{-\frac{\gamma}{N}A_{N_T}(I)} \mathbf{1}_{\{I \in \tilde{B}_{N_T,\Delta N^{1/3}}^+\}} \mathbf{1}_{\{I_{N_T} \leq 0\}}] \\ &= \mathbf{P}_{\beta,0}(I \in \tilde{B}_{N_T,\Delta N^{1/3}}^+) \mathbf{E}_{\beta,0}[e^{-\frac{\gamma}{N}A_{N_T}(I)} \mathbf{1}_{\{I_{N_T} \leq 0\}} | I \in \tilde{B}_{N_T,\Delta N^{1/3}}^+]. \end{aligned}$$

By using the FKG inequality described in Proposition B.1 with $A := \tilde{B}_{N_T,\Delta N^{1/3}}^+$ we obtain

$$(4.59) \quad \begin{aligned} \mathbf{E}_{\beta,0}[e^{-\frac{\gamma}{N}A_{N_T}(I)} \mathbf{1}_{\{I_{N_T} \leq 0\}} | I \in \tilde{B}_{N_T,\Delta N^{1/3}}^+] &\geq \\ \mathbf{E}_{\beta,0}[e^{-\frac{\gamma}{N}A_{N_T}(I)} | I \in \tilde{B}_{N_T,\Delta N^{1/3}}^+] \mathbf{P}_{\beta,0}[I_{N_T} \leq 0 | I \in \tilde{B}_{N_T,\Delta N^{1/3}}^+]. \end{aligned}$$

By Donsker’s invariance principle the rescaled process $(\frac{1}{N^{1/3}} I_{\lfloor sN^{2/3} \rfloor})_{s \in [0, T]}$ converges in distribution towards $(B_s)_{s \in [0, T]}$ a standard Brownian motion, in the set of cadlag functions on $[0, T]$ endowed with the uniform convergence topology. Therefore, we set $m_T := \min\{B_s, s \in [0, T]\}$ and we obtain

$$(4.60) \quad \liminf_{N \rightarrow \infty} V_{N_T} \geq \mathbb{P}\left[B_T < 0, m_T > -\frac{\Delta}{2}\right] \mathbb{E}\left[e^{-\gamma \int_0^T B_s ds} | m_T > -\frac{\Delta}{2}\right].$$

Finally, using (4.51), (4.52), (4.57) and (4.60), we can deduce from (4.48) that

$$(4.61) \quad \begin{aligned} \liminf_{N \rightarrow \infty} \frac{1}{N^{1/3}} \log E_N(\gamma) &\geq -\frac{\log C_3}{T} - 3\gamma\Delta + \frac{1}{T} \log \mathbb{P}\left[B_T < 0, m_T > -\frac{\Delta}{2}\right] \\ &\quad + \frac{1}{T} \log \mathbb{E}\left[e^{-\gamma \int_0^T B_s ds} | m_T > -\frac{\Delta}{2}\right]. \end{aligned}$$

By using [19], Proposition 8.1 Chapter 2, we can state that there exists a $c > 0$ such that for T large enough the inequality $\mathbb{P}[B_T < 0, m_T > -\frac{\Delta}{2}] \geq c\Delta^3/T^{3/2}$ holds true for every $\Delta \in (0, 1]$. As mentioned above (4.40), under the conditioning $m_T > -\Delta/2$, the process $(\Delta/2 + B_s)_{s \in [0, T]}$ has the same law as $(\mathcal{M}_{s,T}^{\Delta/2})_{s \in [0, T]}$. As a consequence, after taking $\limsup_{T \rightarrow \infty}$ in the right-hand side in (4.61) we can claim that for $\Delta \in (0, 1]$,

$$(4.62) \quad \liminf_{N \rightarrow \infty} \frac{1}{N^{1/3}} \log E_N(\gamma) \geq -3\gamma\Delta + \limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{E}[e^{-\gamma \int_0^T \mathcal{M}_{s,T}^{\Delta/2} ds}].$$

We conclude by using Claim 4 and by taking $\limsup_{\Delta \rightarrow 0^+}$ in the right-hand side in (4.62). This proves (4.46) and completes the proof of (4.8). \square

PROOF OF CLAIM 1. Let $\Delta, T > 0$. First we notice that we only need to prove a lower bound on $G_{N,n}^{0,x}$ uniform in $x \in \mathcal{O}_{\Delta/2,\Delta,N}$ as $N \rightarrow \infty$. Indeed, a time-reversal argument yields that $G_{N,n}^{x,0} = G_{N,n}^{0,x}$ for all $x \in \mathbb{N}_0, n \in \mathbb{N}$ (recall that \mathbf{P}_β is symmetric, see (2.8)), and

$$(4.63) \quad \inf_{x \in \mathcal{O}_{\Delta/2,\Delta,N}} G_{N,n}^{0,x} \leq \tilde{G}_{N,\Delta,n}^0 \leq 1.$$

Moreover for any $x \in \mathcal{O}_{\Delta/2,\Delta,N}$ and $\rho \in [1, 3]$ such that $\rho N_T \in \mathbb{N}$, we write

$$(4.64) \quad G_{N,\rho N_T}^{x,0} = \mathbf{E}_{\beta,x}[e^{-\frac{\gamma}{N}A_{\rho N_T}(I)} | I \in B_{\rho N_T}^{0,+}] \mathbf{P}_{\beta,x}(B_{\rho N_T}^{0,+}).$$

Lemma 4.10 claims that the first factor is nonincreasing in x . Moreover, the second factor is polynomial in N uniformly in $x \in \mathcal{O}_{\Delta/2, \Delta, N}$ and $\rho \in [1, 3] \cap \frac{1}{N_T} \mathbb{N}$, see (6.1) below. Thereby, we deduce from (4.64) that

$$(4.65) \quad G_{N, \rho N_T}^{x, 0} \geq \frac{C_{16}}{N^{2/3}} \mathbf{E}_{\beta, \Delta N^{1/3}} [e^{-\frac{\gamma}{N} A_{\rho N_T}(I)} | I \in B_{\rho N_T}^{0,+}],$$

for $N \in \mathbb{N}$, uniformly in $x \in \mathcal{O}_{\Delta/2, \Delta, N}$ and $\rho \in [1, 3] \cap \frac{1}{N_T} \mathbb{N}$ —notice that we wrote $\Delta N^{1/3}$ instead of $\lceil \Delta N^{1/3} \rceil$, and we omit all ceil functions henceforth to lighten notation. Recalling that $\frac{1}{2} T N^{2/3} \leq \rho N_T \leq 3 T N^{2/3}$ and $\rho \in [1, 3]$, (4.65) implies

$$(4.66) \quad G_{N, \rho N_T}^{x, 0} \geq \frac{C_{16}}{N^{2/3}} \mathbf{E}_{\beta, \Delta \sqrt{\frac{2}{T}} \sqrt{\rho N_T}} [e^{-\gamma (\frac{3T}{\rho N_T})^{3/2} A_{\rho N_T}(I)} | I \in B_{\rho N_T}^{0,+}].$$

Furthermore, it is proven in [3], Theorem 2.4, that a properly rescaled random walk of length n starting from $c\sqrt{n}$, $c \in \mathbb{R}_+$, conditioned to remain nonnegative and end in 0, converges in distribution as $n \rightarrow \infty$ to a Brownian bridge starting from c , conditioned to remain nonnegative and to end in 0. Moreover, $\rho N_T \geq \frac{1}{2} T N^{2/3}$ for all $\rho \in [1, 3]$, so the expectation in the r.h.s of (4.66) converges as $N \rightarrow \infty$ to some positive constant, uniformly in $\rho \in [1, 3]$. Recollecting (4.63), this concludes the proof of the claim. \square

PROOF OF CLAIM 2. Recall (4.49) and note that

$$(4.67) \quad \tilde{G}_{N, \Delta, n}^x = \mathbf{E}_{\beta, x} [e^{-\frac{\gamma}{N} A_n(I)} \mathbf{1}_{\{I \in \tilde{B}_{n,0}^+\}} \mathbf{1}_{\{I_n \in \mathcal{O}_{\Delta/2, \Delta, N}\}}].$$

Let us first consider the case $x \in \mathcal{O}_{\frac{\Delta}{2}, \frac{3\Delta}{4}, N}$. We observe that if a trajectory $I := (I_i)_{i=0}^n$ satisfies $I_0 = 0$, $I \in \tilde{B}_{n, \Delta N^{1/3}}^+$ and $I_n \in \mathcal{O}_{0, \Delta/4, N}$ then $x + I := (x + I_i)_{i=0}^n$ satisfies $x + I \in \tilde{B}_{n,0}^+$ and $x + I_n \in \mathcal{O}_{\frac{\Delta}{2}, \Delta, N}$. As a consequence,

$$(4.68) \quad \begin{aligned} \tilde{G}_{N, \Delta, n}^x &\geq \mathbf{E}_{\beta, 0} [e^{-\frac{\gamma}{N} A_n(x+I)} \mathbf{1}_{\{I \in \tilde{B}_{n, \Delta N^{1/3}}^+\}} \mathbf{1}_{\{I_n \in \mathcal{O}_{0, \Delta/4, N}\}}] \\ &\geq \exp\left(-\frac{\gamma \Delta n}{N^{2/3}}\right) U_{N, \Delta, n}^3. \end{aligned}$$

The case $x \in \mathcal{O}_{\frac{3\Delta}{4}, \Delta, N}$ is taken care of similarly. The only difference is that we consider $I := (I_i)_{i=0}^n$ satisfies $I_0 = 0$, $I \in \tilde{B}_{n, \Delta N^{1/3}}^+$ and $I_n \in \mathcal{O}_{-\frac{\Delta}{4}, 0, N}$ such that $x + I \in \tilde{B}_{n,0}^+$ and $x + I_n \in \mathcal{O}_{\frac{\Delta}{2}, \Delta, N}$. Then

$$(4.69) \quad \begin{aligned} \tilde{G}_{N, \Delta, n}^x &\geq \mathbf{E}_{\beta, 0} [e^{-\frac{\gamma}{N} A_n(x+I)} \mathbf{1}_{\{I \in \tilde{B}_{n, \Delta N^{1/3}}^+\}} \mathbf{1}_{\{I_n \in \mathcal{O}_{-\frac{\Delta}{4}, 0, N}\}}] \\ &\geq \exp\left(-\frac{\gamma \Delta n}{N^{2/3}}\right) U_{N, \Delta, n}^2, \end{aligned}$$

and this completes the proof of the claim. \square

PROOF OF CLAIM 3. Let us start with proving the Claim for $j = 2$. We decompose $U_{N, \Delta, n}^2$ depending on the time τ at which the trajectory is above the x -axis for the last time before time n , that is $\tau := \max\{i \leq n : I_i \geq 0\}$. This gives

$$(4.70) \quad U_{N, \Delta, n}^2 = \sum_{k=0}^{n-1} U_{N, \Delta, n}^{2,k} := \sum_{k=0}^{n-1} \mathbf{E}_{\beta, 0} [e^{-\frac{\gamma}{N} A_n(I)} \mathbf{1}_{\{I \in \tilde{B}_{n, \Delta N^{1/3}}^+\}} \mathbf{1}_{\{\tau=k\}} \mathbf{1}_{\{I_n \in \tilde{\mathcal{O}}_{\Delta, N}^2\}}].$$

For $k \in \{0, \dots, n - 1\}$ we partition the trajectories contributing to $U_{N,\Delta,n}^{2,k}$ depending on the values z and y taken by I_τ and $I_{\tau+1}$, respectively. This gives

$$\begin{aligned}
 U_{N,\Delta,n}^{2,k} &:= \sum_{z \geq 0} \sum_{y=1}^{\Delta N^{1/3}/2} \mathbf{E}_{\beta,0} \left[e^{-\frac{\gamma}{N} A_k(I)} \mathbf{1}_{\{I \in \tilde{B}_{k,\Delta N^{1/3}}^+\}} \mathbf{1}_{\{I_k=z\}} \right] \mathbf{P}_\beta(I_1 = -z - y) \\
 (4.71) \quad &\mathbf{E}_{\beta,-y} \left[e^{-\frac{\gamma}{N} A_{n-k-1}(I)} \mathbf{1}_{\{-\frac{\Delta N^{1/3}}{2} \leq I_i < 0, i \leq n-k-1\}} \mathbf{1}_{\{I_{n-k-1} \in \tilde{\mathcal{O}}_{\Delta,N}^2\}} \right].
 \end{aligned}$$

We observe that $\mathbf{P}_\beta(I_1 = -z - y) = c_\beta \mathbf{P}_\beta(I_1 = -z) \mathbf{P}_\beta(I_1 = y)$. Since the increments of I have a symmetric law, we can rewrite the last expectation of the right-hand side in (4.71) as

$$\begin{aligned}
 &\mathbf{E}_{\beta,y} \left[e^{-\frac{\gamma}{N} A_{n-k-1}(-I)} \mathbf{1}_{\{-\frac{\Delta N^{1/3}}{2} \leq -I_i < 0, i \leq n-k-1\}} \mathbf{1}_{\{-I_{n-k-1} \in \tilde{\mathcal{O}}_{\Delta,N}^2\}} \right] \\
 (4.72) \quad &\leq \exp\left(\frac{\gamma \Delta n}{N^{2/3}}\right) \mathbf{E}_{\beta,y} \left[e^{-\frac{\gamma}{N} A_{n-k-1}(I)} \mathbf{1}_{\{0 < I_i \leq \frac{\Delta N^{1/3}}{2}, i \leq n-k-1\}} \mathbf{1}_{\{I_{n-k-1} \in \tilde{\mathcal{O}}_{\Delta,N}^3 \setminus \{0\}\}} \right],
 \end{aligned}$$

where we have used that any trajectory $I = (I_i)_{i=0}^{n-k-1}$ that contributes the expectation of the left-hand side in (4.72) satisfies $A_{n-k-1}(I) = -A_{n-k-1}(-I) \leq \frac{1}{2} \Delta N^{1/3} (n - k - 1)$. Thus, combining (4.71) with (4.72) we obtain

$$\begin{aligned}
 U_{N,\Delta,n}^{2,k} &\leq c_\beta \exp\left(\frac{\gamma \Delta n}{N^{2/3}}\right) \sum_{z \geq 0} \sum_{y=1}^{\Delta N^{1/3}/2} \mathbf{E}_{\beta,0} \left[e^{-\frac{\gamma}{N} A_k(I)} \mathbf{1}_{\{I \in \tilde{B}_{k,\Delta N^{1/3}}^+\}} \mathbf{1}_{\{I_k=z\}} \right] \mathbf{P}_\beta(I_1 = -z) \\
 &\quad \times \mathbf{P}_\beta(I_1 = y) \mathbf{E}_{\beta,y} \left[e^{-\frac{\gamma}{N} A_{n-k-1}(I)} \mathbf{1}_{\{0 < I_i \leq \frac{\Delta N^{1/3}}{2}, i \leq n-k-1\}} \mathbf{1}_{\{I_{n-k-1} \in \tilde{\mathcal{O}}_{\Delta,N}^3 \setminus \{0\}\}} \right] \\
 (4.73) \quad &\leq c_\beta \exp\left(\frac{\gamma \Delta n}{N^{2/3}}\right) \mathbf{E}_{\beta,0} \left[e^{-\frac{\gamma}{N} A_{n+1}(I)} \mathbf{1}_{\{I \in \tilde{B}_{n+1,\Delta N^{1/3}}^+\}} \mathbf{1}_{\{\tilde{\tau}=k+1\}} \mathbf{1}_{\{I_{n+1} \in \tilde{\mathcal{O}}_{\Delta,N}^3\}} \right],
 \end{aligned}$$

where $\tilde{\tau} := \max\{0 \leq j \leq n + 1 : I_j = 0\}$ is the last time before time $n + 1$ at which I touches the x -axis. Therefore,

$$\begin{aligned}
 \sum_{k=0}^{n-1} U_{N,\Delta,n}^{2,k} &\leq c_\beta \exp\left(\frac{\gamma \Delta n}{N^{2/3}}\right) \mathbf{E}_{\beta,0} \left[e^{-\frac{\gamma}{N} A_{n+1}(I)} \mathbf{1}_{\{I \in \tilde{B}_{n+1,\Delta N^{1/3}}^+\}} \mathbf{1}_{\{I_{n+1} \in \tilde{\mathcal{O}}_{\Delta,N}^3\}} \right] \\
 &\leq c_\beta \exp\left(\frac{\gamma \Delta n}{N^{2/3}}\right) U_{N,\Delta,n+1}^3,
 \end{aligned}$$

and this completes the proof of the claim for $j = 2$.

For the case $j = 1$, we assume for notational convenience that $\frac{\Delta N^{1/3}}{4} \in \mathbb{N}$. The method is very close to what we just exposed when $j = 2$. First, we modify the definition of τ in such a way that it becomes the last time before n at which the random walk is not smaller than $-\frac{\Delta N^{1/3}}{4}$, that is, $\tau := \max\{i \leq n : I_i \geq -\frac{\Delta N^{1/3}}{4}\}$. We denote by $-\frac{\Delta N^{1/3}}{4} + z$ the value of I_τ and by $-\frac{\Delta N^{1/3}}{4} - y$ the value of $I_{\tau+1}$ (note that $z \geq 0$ and $1 \leq y < \frac{\Delta N^{1/3}}{4}$). Then, we mimic (4.71) by decomposing each trajectory contributing $U_{N,\Delta,n}^1$ into a first piece between times 0 and τ , a step downwards that equals $-z - y$ and a second piece between times $\tau + 1$ and n . At the cost of a constant factor we transform the step in-between the first and the second piece into two steps of length $-z$ and y , respectively. Finally, we replace the second piece of trajectory by its symmetric with respect to $-\frac{\Delta N^{1/3}}{4}$. Note that this latter transformation lowers the algebraic area swept by this second piece by at most $\frac{\Delta n N^{1/3}}{2}$. Concatenating the first piece, the two steps and the last piece we have built an injective mapping of those trajectories

contributing $U_{N,\Delta,n}^1$ onto those contributing $U_{N,\Delta,n+1}^2$. As for the case $j = 2$, this mapping has an energetic cost (since we keep taking into account the exponential term $-\gamma A_n/N$) that is bounded from above by $\gamma \Delta n/N^{2/3}$, which is sufficient to complete the proof. \square

PROOF OF CLAIM 4.. For conciseness we set, for $\Delta, T > 0$,

$$(4.74) \quad R_\Delta(T) := \mathbb{E}[e^{-\gamma \int_0^T \mathcal{M}_{s,T}^\Delta ds}].$$

We assume by contradiction that there exists an $\varepsilon > 0$ and a $\Delta_0 \in (0, 1]$ such that for every $\Delta \in (0, \Delta_0]$

$$(4.75) \quad \limsup_{T \rightarrow \infty} \frac{1}{T} \log R_\Delta(T) \leq J(\gamma) - \varepsilon.$$

For $\Delta, T > 0$, we set $\tau_\Delta := \inf\{s \geq 0 : \mathcal{M}_{s,T}^0 = \Delta\}$ if $\{s \geq 0 : \mathcal{M}_{s,T}^0 = \Delta\} \neq \emptyset$ and $\tau_\Delta = \infty$ otherwise (τ_Δ depends on T as well but we omit it for conciseness). We state a small ball inequality that is proven at the end of the present proof, that is, there exists $c_1, c_2 > 0$ such that for $T > 0, \eta \in (0, 1]$ and $\Delta > 0$,

$$(4.76) \quad \begin{aligned} \mathbb{P}(\tau_\Delta \geq \eta T) &= \mathbb{P}\left[\max_{s \in [0, \eta T]} \mathcal{M}_{s,T}^0 \leq \Delta\right] \\ &= \mathbb{P}\left[\max_{s \in [0, \eta]} \mathcal{M}_{s,1}^0 \leq \Delta/\sqrt{T}\right] \leq \frac{c_1}{\sqrt{\eta}} e^{-\frac{c_2}{\Delta^2} \eta T}, \end{aligned}$$

where we have used a standard scaling property of Brownian meander to write the second equality in (4.76).

At this stage, we choose $\eta > 0$ and $\Delta \in (0, \Delta_0]$ such that

$$(4.77) \quad \left(J(\gamma) - \frac{\varepsilon}{4}\right)(1 - \eta) \leq J(\gamma) - \frac{\varepsilon}{8}, \quad \text{and} \quad -\frac{c_2 \eta}{\Delta^2} \leq J(\gamma) - \frac{\varepsilon}{2}.$$

Applying Markov property at time τ_Δ we obtain

$$(4.78) \quad \mathbb{E}[e^{-\gamma \int_0^T \mathcal{M}_{s,T}^0 ds}] \leq \mathbb{P}(\tau_\Delta \geq \eta T) + \mathbb{E}[\mathbf{1}_{\{\tau_\Delta < \eta T\}} R_\Delta(T - \tau_\Delta)].$$

Since $\Delta \in (0, \Delta_0]$ we apply (4.75) and there exists a $T_0 > 0$ such that $R_\Delta(T) \leq e^{(J(\gamma) - \varepsilon/4)T}$ for $T \geq T_0$. It remains to apply (4.78) with $T \geq T_0/(1 - \eta)$ in combination with (4.76) and (4.77) to obtain

$$(4.79) \quad \begin{aligned} \mathbb{E}[e^{-\gamma \int_0^T \mathcal{M}_{s,1}^0 ds}] &\leq \frac{c_1}{\sqrt{\eta}} e^{(J(\gamma) - \frac{\varepsilon}{2})T} + \mathbb{E}[\mathbf{1}_{\{\tau_\Delta < \eta T\}} R_\Delta(T - \tau_\Delta)] \\ &\leq \frac{c_1}{\sqrt{\eta}} e^{(J(\gamma) - \frac{\varepsilon}{2})T} + \mathbb{E}[\mathbf{1}_{\{\tau_\Delta < \eta T\}} e^{(J(\gamma) - \varepsilon/4)(T - \tau_\Delta)}] \\ &\leq \frac{c_1}{\sqrt{\eta}} e^{(J(\gamma) - \frac{\varepsilon}{2})T} + e^{(J(\gamma) - \varepsilon/8)T}. \end{aligned}$$

Taking $\frac{1}{T}$ log on both sides in (4.79) and letting $T \rightarrow \infty$, we obtain the contradiction $J(\gamma) \leq J(\gamma) - \varepsilon/8$. This completes the proof of the claim.

Let us quickly sketch the proof of the inequality in (4.76). We use that $(\mathcal{M}_{s,1}^0)_{s \in [0,1]}$ is the limit in distribution of $(B_s)_{s \in [0,1]}$ conditioned on $\{m_1 > -u\}$ as $u \rightarrow 0^+$ where we define $m_t := \min_{s \in [0,t]} (B_s)$ for $0 \leq t \leq 1$ (see [8], Section 2). Therefore, for $\Delta > 0, u \in (0, \Delta)$ and $\eta \in (0, 1)$,

$$(4.80) \quad \mathbb{P}\left[\max_{s \in [0, \eta]} \mathcal{M}_{s,1}^0 \leq \Delta\right] = \lim_{u \rightarrow 0^+} \mathbb{P}\left[\max_{s \in [0, \eta]} B_s \leq \Delta | m_1 > -u\right].$$

By applying Markov property at time $\eta/2$, we can bound from above the probability in the right-hand side in (4.80) by

$$(4.81) \quad \frac{1}{\mathbb{P}(m_1 > -u)} \mathbb{P}(B_s \in (-u, \Delta], \forall s \in [0, \eta/2]) \sup_{x \in (-u, \Delta]} \mathbb{P}_x(B_s \in (-u, \Delta], \forall s \in [0, \eta/2]),$$

and since a Brownian motion $(B_s)_{s \geq 0}$ of law \mathbb{P}_x has the same law as $(x + B_s)_{s \geq 0}$ with $(B_s)_{s \geq 0}$ of law \mathbb{P}_0 , (4.81) is also smaller than

$$(4.82) \quad \frac{\mathbb{P}(m_{\eta/2} > -u)}{\mathbb{P}(m_1 > -u)} \mathbb{P}\left[\max_{s \in [0, \eta/2]} |B_s| \leq 2\Delta\right].$$

At this stage, the inequality in the right-hand side in (4.76) is obtained by combining (4.82) with the following two results:

$$(4.83) \quad \mathbb{P}(m_t > -u) = \sqrt{\frac{2}{\pi}} \int_0^{u/\sqrt{t}} e^{-\frac{x^2}{2}} dx, \quad \text{for } t, u > 0,$$

and there exist $c_1, c_2 > 0$ such that for every $\kappa > 0$

$$(4.84) \quad \mathbb{P}\left(\max_{s \in [0, 1]} |B_s| \leq \kappa\right) \leq c_1 e^{-c_2/\kappa^2}.$$

Note that (4.83) is obtained by observing that, since B is symmetric, $\mathbb{P}(m_t > -u) = \mathbb{P}(M_t < u)$, where M_t is the maximum of B on $[0, t]$ and the law of M_t is well known (see, e.g. [19], Proposition 8.1). Proving (4.84) can be done by estimating the probability that $|B|$ is smaller than κ at times $\{j/k, 1 \leq j \leq k\}$ with $k = \lfloor 1/\kappa^2 \rfloor$ and by applying Markov property at those times. \square

4.5. *Step 5: Horizontal extension inside the collapsed phase.* We now prove that in the collapsed phase, the typical horizontal extension of the polymer is of order \sqrt{L} for L large. For any interval $I \subseteq \mathbb{R}$, define

$$(4.85) \quad \tilde{Z}_{L, \beta, \delta}^{\text{bead}}(I) = \frac{1}{c_\beta e^{\beta L}} Z_{L, \beta, \delta}^{\text{bead}}(I) := \sum_{\substack{1 \leq N \leq L/2, \\ N \in I \cap \mathbb{N}}} (\Gamma_\beta)^{2N} D_{N, q_N^L}^{\text{bead}},$$

where q_N^L and $D_{N, q_N^L}^{\text{bead}}$ are defined in Proposition 4.1. Recall that $\mathcal{C}_{\text{bead}}$ denotes the collapsed phase of the single-bead model (see Proposition 3.1), and let $(\mathcal{C}_{\text{bead}})^\circ$ denote its interior.

LEMMA 4.11. *Let $(\beta, \delta) \in (\mathcal{C}_{\text{bead}})^\circ$, there exists $(a_1, a_2) \in (0, \infty)^2$ such that*

$$(4.86) \quad \lim_{L \rightarrow \infty} \frac{Z_{L, \beta, \delta}^{\text{bead}}([a_1, a_2]\sqrt{L})}{Z_{L, \beta, \delta}^{\text{bead}}} = 1.$$

PROOF. Recall (4.7) and (4.8). By relaxing all constraints on S, I but $\{I \in B_N^{0,+}\}$ in (4.8), one has the obvious bound $D_{N, q}^{\text{bead}} \leq Z_{\text{wet}, N}^{\beta, \delta}$ (recall (A.1)). Moreover, Proposition A.1 (ii) implies that there exists some $K > 0$ such that $Z_{\text{wet}, N}^{\beta, \delta} \leq K e^{h_\beta(\delta)N}$ for $N \geq 1$. Thereby,

$$\tilde{Z}_{L, \beta, \delta}^{\text{bead}}([a_2\sqrt{L}, L]) = \sum_{N=a_2\sqrt{L}}^{L/2} (\Gamma_\beta)^{2N} D_{N, q_N^L}^{\text{bead}} \leq \sum_{N=a_2\sqrt{L}}^{L/2} (\Gamma_\beta)^{2N} K e^{h_\beta(\delta)N}.$$

In $(\mathcal{C}_{\text{bead}})^\circ$, one has $2 \log \Gamma_\beta + h_\beta(\delta) < 0$ (recall (3.6)), so we conclude that there exist $c_1, c_2 > 0$ such that,

$$(4.87) \quad \tilde{Z}_{L, \beta, \delta}^{\text{bead}}([a_2\sqrt{L}, L]) \leq c_1 e^{-c_2 a_2 \sqrt{L}},$$

for all $L \geq 1$ and $a_2 \in \frac{\mathbb{N}}{\sqrt{L}}$.

Regarding trajectories with an horizontal extension smaller than $a_1\sqrt{L}$, we use $\Gamma_\beta \leq 1$ and (4.7)–(4.8) to write

$$(4.88) \quad \tilde{Z}_{L,\beta,\delta}^{\text{bead}}([1, a_1\sqrt{L}]) \leq e^{\delta a_1\sqrt{L}} \sum_{N \leq a_1\sqrt{L}} \mathbf{P}_\beta \left(A_{N+1}(S) \geq \frac{L}{2} - N \right).$$

The inequality $A_{N+1}(S) \geq \frac{L}{2} - N$ implies that $\max\{S_i, i \leq N\} \geq \frac{L/2-N}{N+1}$. Therefore, for L large enough we can assert that for every $a_1 \leq 1$ and $N \leq a_1\sqrt{L}$

$$(4.89) \quad \mathbf{P}_\beta \left(A_{N+1}(S) \geq \frac{L}{2} - N \right) \leq \mathbf{P}_\beta \left(\sum_{i \leq a_1\sqrt{L}} |S_i - S_{i-1}| \geq \frac{\sqrt{L}}{4a_1} \right).$$

Since $|S_1 - S_0|$ (of law \mathbf{P}_β) has finite small exponential moments, Chernov’s bound guarantees us that there exists a function $\rho : (0, \infty) \mapsto \mathbb{R}^+$ satisfying $\lim_{a_1 \rightarrow 0^+} \rho(a_1) = \infty$ such that the right-hand side in (4.89) is bounded above by $e^{-\rho(a_1)\sqrt{L}}$. Therefore, for L large enough and $a_1 \leq 1$,

$$(4.90) \quad \tilde{Z}_{L,\beta,\delta}^{\text{bead}}([1, a_1\sqrt{L}]) \leq \sqrt{L} e^{-(\rho(a_1) - \delta a_1)\sqrt{L}}.$$

At this stage it remains to display a lower bound on $Z_{L,\beta,\delta}^{\text{bead}}$. To that aim, we only consider the term indexed by $N = \sqrt{L}$ in the sum in (4.7). Applying Proposition 4.2 with $q = \frac{1}{2} - \frac{1}{\sqrt{L}}$ and using that g is C^1 we deduce that there exists a $C > 0$ such that

$$(4.91) \quad \tilde{Z}_{L,\beta,\delta}^{\text{bead}} \geq \frac{C}{L} (\Gamma_\beta)^{2\sqrt{L}} e^{-g(\frac{1}{2}, 0)\sqrt{L}} E_{\sqrt{L}, \frac{1}{2} - \frac{1}{\sqrt{L}}}^{\text{bead}}.$$

It remains to bound $E_{\sqrt{L}, \frac{1}{2} - \frac{1}{\sqrt{L}}}^{\text{bead}}$ from below by choosing the trajectory $(I_i)_{i=0}^{\sqrt{L}}$ that sticks to zero, that is, $E_{\sqrt{L}, \frac{1}{2} - \frac{1}{\sqrt{L}}}^{\text{bead}} \geq (e^\delta/c_\beta)^{\sqrt{L}}$. We set $\kappa := 2 \log \Gamma_\beta + \delta - g(1/2, 0) - \log c_\beta$ and we obtain that for L large enough

$$(4.92) \quad \tilde{Z}_{L,\beta,\delta}^{\text{bead}} \geq \frac{C}{L} e^{\kappa\sqrt{L}}.$$

Combining (4.87), (4.90) and (4.92), it suffices to choose a_1 (resp. a_2) small enough (resp. large enough) for (4.86) to hold. \square

4.6. *Step 6: Proof of Theorem 3.2.* We finally have all required estimates to prove Theorem 3.2. Assume $\beta > \beta_c$ and $\delta < \delta_c^{\text{bead}}(\beta)$. Applying Proposition 4.1 and Lemma 4.11, we can restrict the sum in (4.7) to $N \in [a_1, a_2]\sqrt{L} \cap \mathbb{N}$. In particular, this implies that $q_N^L = \frac{L-2N}{2N^2} \in [\frac{1}{2a_2^2} - \frac{1}{a_1\sqrt{L}}, \frac{1}{2a_1^2}]$, and $\frac{1}{2a_2^2} - \frac{1}{a_1\sqrt{L}} > 0$ for L sufficiently large. Thereby Proposition 4.2 yields

$$(4.93) \quad Z_{L,\beta,\delta}^{\text{bead}} \asymp \frac{e^{\beta L}}{L} \sum_{N=a_1\sqrt{L}}^{a_2\sqrt{L}} (\Gamma_\beta)^{2N} e^{-Ng(\frac{L}{2N^2}, 0)} E_{N, q_N^L}^{\text{bead}},$$

uniformly in $L \in \mathbb{N}$ sufficiently large, and where we also used that g is C^1 , so $g(q_N^L, 0) = g(\frac{L}{2N^2}, 0) + O(1/N)$ uniformly in $N \in [a_1, a_2]\sqrt{L} \cap \mathbb{N}$.

Case $\tilde{\delta}_c(\beta) < \delta < \delta_c^{\text{bead}}(\beta)$. We apply Proposition 4.6 to (4.93), to write

$$(4.94) \quad Z_{L,\beta,\delta}^{\text{bead}} \asymp \frac{e^{\beta L}}{L} \sum_{N=a_1\sqrt{L}}^{a_2\sqrt{L}} e^{\sqrt{L}\varphi_{\beta,\delta}(\frac{N}{\sqrt{L}})},$$

where we recall the definition of $\varphi_{\beta,\delta}$ from (3.13). For notational convenience, we omit the subscript (β, δ) of φ in what follows. Recall that φ is C^2 and strictly concave. Moreover, for $(\beta, \delta) \in (\mathcal{C}_{\text{bead}})^0$ φ is negative on $(0, \infty)$ and reaches its maximum at some $\tilde{a} = \tilde{a}(\beta, \delta) \in (0, \infty)$. Therefore, provided that a_1 (resp. a_2) is sufficiently small (resp. large), we can assume $\tilde{a} \in (a_1, a_2)$. Because φ is strictly concave and C^2 on $[a_1, a_2]$, there are constants $c, c' > 0$ such that

$$(4.95) \quad -c(a - \tilde{a})^2 \leq \varphi(a) - \varphi(\tilde{a}) \leq -c'(a - \tilde{a})^2,$$

for all $a \in [a_1, a_2]$. Thus we can rewrite (4.94) as

$$(4.96) \quad \begin{aligned} & C_{17} \frac{e^{\beta L}}{L} e^{\sqrt{L}\varphi(\tilde{a})} \sum_{N=a_1\sqrt{L}}^{a_2\sqrt{L}} e^{-c\sqrt{L}(\frac{N}{\sqrt{L}} - \tilde{a})^2} \\ & \leq Z_{L,\beta,\delta}^{\text{bead}} \leq C_{18} \frac{e^{\beta L}}{L} e^{\sqrt{L}\varphi(\tilde{a})} \sum_{N=a_1\sqrt{L}}^{a_2\sqrt{L}} e^{-c'\sqrt{L}(\frac{N}{\sqrt{L}} - \tilde{a})^2}. \end{aligned}$$

Finally, we observe that for any $c > 0$ and $b_1 < 0 < b_2$, one has

$$(4.97) \quad \sum_{\substack{b_1\sqrt{L} \leq N \leq b_2\sqrt{L}, \\ N \in \mathbb{Z}}} e^{-c\frac{N^2}{L}} \asymp L^{1/4}.$$

Indeed, let $R > 0$, and define $I_{R,L} = [-RL^{1/4}, RL^{1/4}]$. We decompose the above sum into $S_1^{L,\beta,R} + S_2^{L,\beta,R}$, where

$$S_1^{L,\beta,R} := \sum_{N \in I_{R,L} \cap \mathbb{Z}} e^{-c\frac{N^2}{L}}, \quad \text{and} \quad S_2^{L,\beta,R} := \sum_{\substack{N \in [b_1\sqrt{L}, b_2\sqrt{L}] \setminus I_{R,L}, \\ N \in \mathbb{Z}}} e^{-c\frac{N^2}{L}}.$$

Let us first handle $S_1^{L,\beta,R}$. With a Riemann sum approximation, we have

$$(4.98) \quad L^{-1/4} S_1^{L,\beta,R} = L^{-1/4} \sum_{N=-RL^{1/4}}^{RL^{1/4}} e^{-c(\frac{N}{L^{1/4}})^2} \xrightarrow{L \rightarrow \infty} \int_{-R}^R e^{-cx^2} dx,$$

therefore $S_1^{L,\beta,R} \sim L^{1/4} \int_{-R}^R e^{-cx^2} dx$ as $L \rightarrow \infty$. Regarding $S_2^{L,\beta,R}$, we have

$$\sum_{\substack{N \in [b_1\sqrt{L}, b_2\sqrt{L}] \setminus I_{R,L}, \\ N \in \mathbb{Z}}} e^{-c\frac{N^2}{L}} \leq 2 \sum_{N > RL^{1/4}} e^{-c(\frac{N}{L^{1/4}})^2},$$

and a Riemann sum approximation yields that

$$(4.99) \quad \lim_{L \rightarrow \infty} L^{-1/4} \sum_{N > RL^{1/4}} e^{-c(\frac{N}{L^{1/4}})^2} = \int_R^\infty e^{-cx^2} dx.$$

Hence there is $C_R > 0$ such that $S_2^{L,\beta,R} \leq C_R L^{1/4}$ for all $L \geq 1$. Together with (4.98), this concludes the proof of (4.97). Plugging (4.97) into (4.96), this proves Theorem 3.2 in $(\mathcal{AC})^0$.

Case $\delta = 0$. We now prove the theorem when $\delta = 0$ —then the case $0 < \delta \leq \tilde{\delta}_c(\beta)$ is a straightforward consequence of the other two. Let $\varepsilon > 0$. Recollecting (4.93) and Proposition 4.8, and noticing that $q \mapsto J(\sigma_\beta \partial_q g(q, 0))$ is \mathcal{C}^1 on $(0, \infty)$; we can write the bounds

$$(4.100) \quad \begin{aligned} & C_{19} \frac{e^{\beta L}}{L} \sum_{N=a_1\sqrt{L}}^{a_2\sqrt{L}} \exp\left(\sqrt{L}\varphi\left(\frac{N}{\sqrt{L}}\right) + L^{1/6}\left(\psi\left(\frac{N}{\sqrt{L}}\right) - \varepsilon\right)\right) \\ & \leq Z_{L,\beta,0}^{\text{bead}} \leq C_{20} \frac{e^{\beta L}}{L} \sum_{N=a_1\sqrt{L}}^{a_2\sqrt{L}} \exp\left(\sqrt{L}\varphi\left(\frac{N}{\sqrt{L}}\right) + L^{1/6}\left(\psi\left(\frac{N}{\sqrt{L}}\right) + \varepsilon\right)\right), \end{aligned}$$

for L sufficiently large, where φ is defined in (3.13), and we define for all $a \in (0, \infty)$,

$$(4.101) \quad \psi(a) := a^{1/3} J\left(\sigma_\beta \partial_q g\left(\frac{1}{2a^2}, 0\right)\right) < 0.$$

Notice that we can drop the constant and polynomial factors by slightly adjusting ε . Let us fix some $R > 0$ and define

$$I_{R,L}^1 := \tilde{a}\sqrt{L} + [-RL^{1/3}, RL^{1/3}], \quad \text{and} \quad I_{R,L}^2 := [a_1\sqrt{L}, a_2\sqrt{L}] \setminus I_{R,L}^1,$$

so (4.100) yields

$$(4.102) \quad e^{\beta L} Q_{L,R,-\varepsilon}^1 \leq Z_{L,\beta,0}^{\text{bead}} \leq e^{\beta L} (Q_{L,R,\varepsilon}^1 + Q_{L,R,\varepsilon}^2),$$

where we define for any $i \in \{1, 2\}$ and $\eta \in \{-\varepsilon, \varepsilon\}$,

$$Q_{L,R,\eta}^i := \sum_{N \in I_{R,L}^i} \exp\left(\sqrt{L}\varphi\left(\frac{N}{\sqrt{L}}\right) + L^{1/6}\left(\psi\left(\frac{N}{\sqrt{L}}\right) + \eta\right)\right).$$

Regarding the lower bound, recall that ψ is \mathcal{C}^1 on $(0, \infty)$, so there is some $C_{21} > 0$ such that

$$(4.103) \quad \left| \psi\left(\frac{N}{\sqrt{L}}\right) - \psi(\tilde{a}) \right| \leq C_{21} RL^{-1/6},$$

uniformly in L sufficiently large and $N \in I_{R,L}^1$. Hence, for $R \geq 1$ and L large,

$$(4.104) \quad \begin{aligned} Q_{L,R,-\varepsilon}^1 & \geq Q_{L,1,-\varepsilon}^1 \geq \exp(\sqrt{L}\varphi(\tilde{a}) + L^{1/6}(\psi(\tilde{a}) - \varepsilon) - C_{21}) \\ & \times \sum_{N=\tilde{a}\sqrt{L}-L^{1/3}}^{\tilde{a}\sqrt{L}+L^{1/3}} \exp\left(\sqrt{L}\left(\varphi\left(\frac{N}{\sqrt{L}}\right) - \varphi(\tilde{a})\right)\right), \end{aligned}$$

so that by recalling (4.95) and (4.98) we obtain that the latter sum is of order $C_{22}L^{1/4} \geq 1$ when $L \rightarrow \infty$, which yields the expected lower bound (by slightly adjusting ε).

Similarly to the lower bound just above, we deduce from (4.103) and from (4.95) to (4.98) that

$$(4.105) \quad Q_{L,R,\varepsilon}^1 \leq \exp(\sqrt{L}\varphi(\tilde{a}) + L^{1/6}(\psi(\tilde{a}) + 2\varepsilon)),$$

for L sufficiently large. Regarding $Q_{L,R,\varepsilon}^2$, (4.95) yields that for all $N \in I_{R,L}^2$,

$$\sqrt{L}\left(\varphi\left(\frac{N}{\sqrt{L}}\right) - \varphi(\tilde{a})\right) \leq -c'R^2L^{1/6}.$$

Moreover, $\psi(N/\sqrt{L}) + \varepsilon \leq \frac{1}{2} \sup_{[a_1, a_2]} \psi < 0$ for all $N \in [a_1\sqrt{L}, a_2\sqrt{L}]$, $L \in \mathbb{N}$ provided that ε is sufficiently small. Thereby,

$$Q_{L,R,\varepsilon}^2 \leq (a_2 - a_1)\sqrt{L} \times \exp(\sqrt{L}\varphi(\tilde{a}) - c'R^2L^{1/6}).$$

Finally, noticing that $Q_{L,R,\varepsilon}^1 \geq Q_{L,R,-\varepsilon}^1$ and recalling (4.104), we have $Q_{L,R,\varepsilon}^2/Q_{L,R,\varepsilon}^1 = o(1)$, provided that R satisfies $-c'R^2 < \inf_{[a_1,a_2]} \psi - \varepsilon$. Recollecting (4.102) this completes the proof of the upper bound, and concludes the case $\delta = 0$.

Case $0 < \delta \leq \tilde{\delta}_c(\beta)$. When $\delta \in (0, \tilde{\delta}(\beta)]$, we do not provide sharp estimates but we can still claim the following: first, $\delta \mapsto Z_{L,\beta,\delta}^{\text{bead}}$ is clearly nondecreasing, so $Z_{L,\beta,\delta}^{\text{bead}} \geq Z_{L,\beta,0}^{\text{bead}}$, and the lower bound from the case $\delta = 0$ also holds for any $\delta > 0$. Second, notice that we have $E_{N,q}^{\text{bead}} \leq Z_{\text{wet},N}^{\beta,\delta}$ for all $q > 0$, $N \in \mathbb{N}$ (see (4.10) and (A.1)), so Proposition A.1 (ii) implies that there exists $C_{23} > 0$ (which depends on β, δ) such that $E_{N,q}^{\text{bead}} \leq C_{23}$ for $N \in \mathbb{N}$. Thus (4.93) yields

$$Z_{L,\beta,\delta}^{\text{bead}} \leq \frac{e^{\beta L}}{L} C_{23} \sum_{N=a_1\sqrt{L}}^{a_2\sqrt{L}} e^{\sqrt{L}\varphi(\frac{N}{\sqrt{L}})},$$

where φ is defined in (3.13), with $h_\beta(\delta) = 0$ for all $\delta \in (0, \tilde{\delta}(\beta)]$. Then the behavior of the sum above is the same as in the case $\delta \in (\tilde{\delta}(\beta), \delta_c^{\text{bead}}(\beta))$, which yields the same upper bound. This fully concludes the proof of Theorem 3.2, subject to Lemmas 4.5 and 4.10, and Proposition 4.3.

4.7. Proof of Corollary 3.3. In this section, we prove Corollary 3.3, which follows directly from Theorem 3.2. Let $\beta > \beta_c$. We first focus on the case $\delta \in (\tilde{\delta}_c(\beta), \delta_c^{\text{bead}}(\beta))$. Recall the expression of $\varphi'(a) = \varphi'_{(\beta,\delta)}(a)$ in (3.14) and the definition of $\tilde{a}(\beta, \delta)$, then notice that $(\delta, a) \mapsto \varphi'_{(\beta,\delta)}(a)$ is \mathcal{C}^1 on $(\tilde{\delta}_c(\beta), \delta_c^{\text{bead}}(\beta)) \times (0, +\infty)$ and $\varphi''_{(\beta,\delta)}(\tilde{a}(\beta, \delta)) < 0$. It follows from the implicit function theorem that $\delta \mapsto \tilde{a}(\beta, \delta)$ is \mathcal{C}^1 on $(\tilde{\delta}_c(\beta), \delta_c^{\text{bead}}(\beta))$. Hence, $\delta \mapsto \Phi(\beta, \delta)$ is \mathcal{C}^1 on $(\tilde{\delta}_c(\beta), \delta_c^{\text{bead}}(\beta))$ too.

Let $\delta \in (\tilde{\delta}_c(\beta), \delta_c^{\text{bead}}(\beta))$ and $\varepsilon > 0$, and let us denote for $\ell \in \Omega_L^{\text{bead}}$,

$$Q(\ell) := \sum_{k=1}^{N_\ell} \mathbf{1}_{\{\sum_{i=1}^k \ell_i = 0\}},$$

for conciseness. We recall the definition of $\mathbf{P}_{L,\beta,\delta}^{\text{bead}}$ in (3.4) and we denote by $\mathbf{E}_{L,\beta,\delta}^{\text{bead}}$ its associated expectation. Let $t > 0$ be such that $\delta + t \in (\tilde{\delta}_c(\beta), \delta_c^{\text{bead}}(\beta))$, and write with Chernov's bound

$$\begin{aligned} \mathbf{P}_{L,\beta,\delta}^{\text{bead}}(Q(\ell) \geq (\partial_\delta \Phi(\beta, \delta) + \varepsilon)\sqrt{L}) &\leq \mathbf{E}_{L,\beta,\delta}^{\text{bead}}[e^{tQ(\ell)}] e^{-t(\partial_\delta \Phi(\beta,\delta) + \varepsilon)\sqrt{L}} \\ &\leq \frac{Z_{L,\beta,\delta+t}^{\text{bead}}}{Z_{L,\beta,\delta}^{\text{bead}}} e^{-t(\partial_\delta \Phi(\beta,\delta) + \varepsilon)\sqrt{L}}. \end{aligned}$$

Applying Theorem 3.2(i), we therefore get that there exists some $C_{24} > 0$ such that for $L \geq 1$,

$$\mathbf{P}_{L,\beta,\delta}^{\text{bead}}(Q(\ell) \geq (\partial_\delta \Phi(\beta, \delta) + \varepsilon)\sqrt{L}) \leq C_{24} e^{(\Phi(\beta,\delta+t) - \Phi(\beta,\delta) - t\partial_\delta \Phi(\beta,\delta) - t\varepsilon)\sqrt{L}}.$$

Finally, $\Phi(\beta, \delta + t) - \Phi(\beta, \delta) - t\partial_\delta \Phi(\beta, \delta) = o(t)$ as $t \rightarrow 0$ (since $\delta \mapsto \Phi(\beta, \delta)$ is \mathcal{C}^1), so we can fix $t > 0$ such that the right-hand side above goes to 0 as $L \rightarrow \infty$. Similarly, we can write for $s > 0$ such that $\delta - s \in (\tilde{\delta}_c(\beta), \delta_c^{\text{bead}}(\beta))$,

$$\begin{aligned} \mathbf{P}_{L,\beta,\delta}^{\text{bead}}(Q(\ell) \leq (\partial_\delta \Phi(\beta, \delta) - \varepsilon)\sqrt{L}) &\leq \frac{Z_{L,\beta,\delta-s}^{\text{bead}}}{Z_{L,\beta,\delta}^{\text{bead}}} e^{s(\partial_\delta \Phi(\beta,\delta) - \varepsilon)\sqrt{L}} \\ &\leq C_{25} e^{(\Phi(\beta,\delta-s) - \Phi(\beta,\delta) + s\partial_\delta \Phi(\beta,\delta) - s\varepsilon)\sqrt{L}}, \end{aligned}$$

for some $C_{25} > 0$, where we also applied Chernov’s bound and Theorem 3.2(i). We can then choose $s > 0$ such that this term also goes to 0 as $L \rightarrow \infty$. This concludes the proof of Corollary 3.3(i).

Regarding the case $\delta \in [0, \tilde{\delta}_c(\beta))$, let $K > 0$, $\varepsilon > 0$ and $t > 0$ be such that $\delta + t < \tilde{\delta}_c(\beta)$. Using once again Chernov’s bound and Theorem 3.2(ii), there exist $C_{26} > 0$ and $L_0 \in \mathbb{N}$ such that for $L \geq L_0$,

$$\begin{aligned} \mathbf{P}_{L,\beta,\delta}^{\text{bead}}(Q(\ell) \geq KL^{1/6}) &\leq \mathbf{E}_{L,\beta,\delta}^{\text{bead}}[e^{tQ(\ell)}]e^{-tKL^{1/6}} \\ &\leq \frac{Z_{L,\beta,\delta+t}^{\text{bead}}}{Z_{L,\beta,\delta}^{\text{bead}}}e^{-tKL^{1/6}} \leq C_{26}L^{-3/4}e^{(-\Psi(\beta)+\varepsilon-tK)L^{1/6}}. \end{aligned}$$

This goes to 0 as $L \rightarrow \infty$ provided that K has been fixed sufficiently large, and concludes the proof of Corollary 3.3(ii). \square

5. Proof of Theorem 2.2 and Proposition 3.1.

5.1. *Proof of Theorem 2.2.* The proof of Theorem 2.2 also relies on the random walk representation introduced in [23] and adapted in the proof of Theorem 3.2 (see Section 4), but it is much less involved. We divide the proof into 4 steps. First, we prove that the free energy is not changed when we additionally constrain the polymer to end on the horizontal axis (i.e., $\sum_{i=1}^N \ell_i = 0$)—in particular, the *constrained* partition function $Z_{L,\beta,\delta}^{+,c}$ is super-multiplicative, which implies the well-posedness of f . Next, we adapt to the present model the random-walk representation of IPDSAW. As in Section 4, we derive a probabilistic representation of the partition function by rewriting, for every $N \leq L$, the contribution to the partition function of those trajectories made of N stretches, in terms of two auxiliary (coupled) random walks S and I . Then we compute the generating function of the partition function $Z_{L,\beta,\delta}^{+,c}$. With our random walk representation, we can rewrite it as the partition function of a wetting model for two independent random walks, with the in-between area constraint of S and I becoming an in-between area penalization in the generating function. This allows us to characterize \tilde{f} in terms of the free energy of this “coupled, in-between area penalized” wetting model. Finally we place ourselves on the critical curve between \mathcal{C} and \mathcal{E} where the area penalization vanishes, and we apply well-known results on the wetting model to derive the equation of the curve.

Step 1: Constraining the partition function. Let us define the *constrained partition function*:

$$(5.1) \quad Z_{L,\beta,\delta}^{+,c} := \sum_{N=1}^L \sum_{\substack{\ell \in \mathcal{L}_{N,L}^+ \\ \ell_0 = \ell_{N+1} = 0}} e^{H(\ell)} \mathbf{1}_{\{\sum_{i=1}^N \ell_i = 0\}}.$$

Notice that this partition function is super-multiplicative: indeed for any $L_1, L_2 \geq 0$, we bound $Z_{L_1+L_2,\beta,\delta}^{+,c}$ from below by constraining it to touch the axis between the $(L_1 - 1)$ th and L_1 th monomers; then we notice that the contribution to $H(\ell)$ from self-touching between the two parts of the polymer (before and after the segment $(L_1 - 1, L_1)$) are nonnegative; and we separate the trajectory in two at the L_1 th monomer (recall that all our trajectories end with a horizontal step) to finally write

$$(5.2) \quad Z_{L_1+L_2,\beta,\delta}^{+,c} \geq Z_{L_1,\beta,\delta}^{+,c} Z_{L_2,\beta,\delta}^{+,c}.$$

Hence $\frac{1}{L} \log Z_{L,\beta,\delta}^{+,c}$ converges as $L \rightarrow \infty$, by Fekete’s lemma.

We now claim that $Z_{L,\beta,\delta}^+$ and $Z_{L,\beta,\delta}^{+,c}$ are comparable. More precisely for $L \in \mathbb{N}$,

$$(5.3) \quad \frac{1}{L^2} (Z_{L,\beta,\delta}^+)^2 \leq Z_{2L,\beta,\delta}^{+,c} \leq Z_{2L,\beta,\delta}^+.$$

The upper bound is straightforward. For the lower bound, we constrain $Z_{2L,\beta,\delta}^{+,c}$ to make a horizontal step between the $(L - 1)$ th and L th monomers, and sum over its possible heights. Writing $z_{L-1} = (x_{L-1}, y_{L-1})$ and $z_L = (x_L, y_L)$ the coordinates of those monomers, we have

$$Z_{2L+2,\beta,\delta}^{+,c} \geq \sum_{y=0}^{L-1} Z_{2L,\beta,\delta}^+(y_{L-1} = y_L = y, x_L = 1 + x_{L-1}).$$

We then separate the trajectory at the $(L - 1)$ th monomer. The first half gives the term $Z_{L,\beta,\delta}^+(y_L = y)$ (recall that the last step is always horizontal). As for the second half, we shift the last horizontal step to just before z_L (which costs at most a factor e^δ), then we reverse the trajectory horizontally, to obtain that it is bounded from below by $Z_{L,\beta,\delta}^+(y_L = y)$. Therefore,

$$Z_{2L,\beta,\delta}^{+,c} \geq \sum_{y=0}^{L-1} (Z_{L,\beta,\delta}^+(y_L = y))^2 \geq \left(\sup_{0 \leq y \leq L-1} Z_{L,\beta,\delta}^+(y_L = y) \right)^2,$$

and we conclude the proof of (5.3) by writing $Z_{L,\beta,\delta}^+ \leq L \sup_{0 \leq y \leq L-1} Z_{L,\beta,\delta}^+(y_L = y)$. In particular, this implies that

$$(5.4) \quad f(\beta, \delta) = \lim_{L \rightarrow \infty} \frac{1}{L} \log Z_{L,\beta,\delta}^+ = \lim_{L \rightarrow \infty} \frac{1}{L} \log Z_{L,\beta,\delta}^{+,c},$$

is well defined, and that we can replace $Z_{L,\beta,\delta}^+$ with $Z_{L,\beta,\delta}^{+,c}$ to prove Theorem 2.2.

Step 2: A random-walk representation. We provide a probabilistic description of the constrained partition function $Z_{L,\beta,\delta}^{+,c}$ as in Section 4—which also applies to the free counterpart $Z_{L,\beta,\delta}^+$. Recalling (5.1), (2.2) and (4.1), we can rewrite the constrained partition function similarly to (4.2) to obtain

$$(5.5) \quad Z_{L,\beta,\delta}^{+,c} = e^{\beta L} \sum_{N=1}^L e^{-\beta N} \sum_{\substack{\ell \in \mathcal{L}_{N,L}^+ \\ \ell_0 = \ell_{N+1} = 0}} e^{-\frac{\beta}{2} \sum_{i=0}^N |\ell_i + \ell_{i+1}|} e^{\delta \sum_{k=1}^N \mathbf{1}_{\{\sum_{i=1}^k \ell_i = 0\}}} \mathbf{1}_{\{\sum_{i=1}^N \ell_i = 0\}}.$$

Recall the definition of \mathbf{P}_β in (2.8): similarly to (4.3), we consider two independent one-dimensional random walks $S := (S_i)_{i \geq 0}$ and $I := (I_i)_{i \geq 0}$ starting from 0 and such that $(S_{i+1} - S_i)_{i \geq 0}$ and $(I_{i+1} - I_i)_{i \geq 0}$ are i.i.d. sequences of random variables of law \mathbf{P}_β . We notice that for every $\ell \in \mathcal{L}_{N,L}^+$ (with $\ell_0 = \ell_{N+1} = 0$) the first factor in the second sum in (5.5) satisfies

$$(5.6) \quad e^{-\frac{\beta}{2} \sum_{i=0}^N |\ell_i + \ell_{i+1}|} = c_\beta^{N+1} \mathbf{P}_\beta \left(S_k = \sum_{i=0}^{2k-1} \ell_i, \forall k \leq N_S \right) \mathbf{P}_\beta \left(I_k = \sum_{i=0}^{2k} \ell_i, \forall k \leq N_I \right),$$

with $N_S := \lfloor N/2 \rfloor + 1$ and $N_I := \lceil N/2 \rceil$. Hence, we define S and I as in Section 4.1. Similarly to (4.4), define

$$(5.7) \quad G_N(S, I) := \sum_{k=1}^{N_I} |I_k - S_k| + \sum_{k=1}^{N_S} |S_k - I_{k-1}|,$$

and our definition of S, I implies $\sum_{i=1}^N |\ell_i| = G_N(S, I)$. Finally, the constraint $\sum_{i=1}^N \ell_i = 0$ is equivalent to $S_{N_S} = I_{N_I} = 0$ (recall $\ell_{N+1} = 0$).

Defining $\tilde{Z}_{L,\beta,\delta}^{+,c} := \frac{1}{c_\beta} e^{-\beta L} Z_{L,\beta,\delta}^{+,c}$, and plugging (5.6) into (5.5), we obtain

$$(5.8) \quad \begin{aligned} \tilde{Z}_{L,\beta,\delta}^{+,c} &= \sum_{N=1}^L (\Gamma_\beta)^N \mathbf{E}_\beta [e^{\delta \sum_{k=1}^{N_S} \mathbf{1}_{\{s_k=0\}}} e^{\delta \sum_{k=1}^{N_I} \mathbf{1}_{\{l_k=0\}}} \\ &\quad \times \mathbf{1}_{\{G_N(S,I)=L-N\}} \mathbf{1}_{\{S \in B_{N_S}^{0,+}\}} \mathbf{1}_{\{I \in B_{N_I}^{0,+}\}}], \end{aligned}$$

where we recall $\Gamma_\beta = c_\beta e^{-\beta}$, $G_N(S, I)$ is defined in (5.7), and $B_n^{0,+}$ is the set of nonnegative trajectories of length n ending on the horizontal axis (see (2.10)).

Step 3: The generating function of $\tilde{Z}_{L,\beta,\delta}^{+,c}$. Recalling (5.4), the excess free energy satisfies

$$(5.9) \quad \tilde{f}(\beta, \delta) = \sup \left\{ \gamma \geq 0; \sum_{L \geq 1} \tilde{Z}_{L,\beta,\delta}^{+,c} e^{-\gamma L} = +\infty \right\},$$

and $\tilde{f}(\beta, \delta) = 0$ if this set is empty. Let us compute the generating function of the tilted partition function. Recalling (5.8) and inverting the sums over L and N , we obtain

$$(5.10) \quad \sum_{L \geq 1} \tilde{Z}_{L,\beta,\delta}^{+,c} e^{-\gamma L} = \sum_{N \geq 1} (\Gamma_\beta e^{-\gamma})^N Q_N^{\beta,\delta,\gamma},$$

where we define

$$(5.11) \quad Q_N^{\beta,\delta,\gamma} := \mathbf{E}_{\beta,0} [e^{-\gamma G_N(S,I)} e^{\delta \sum_{k=1}^{N_S} \mathbf{1}_{\{s_k=0\}}} e^{\delta \sum_{k=1}^{N_I} \mathbf{1}_{\{l_k=0\}}} \mathbf{1}_{\{S \in B_{N_S}^{0,+}, I \in B_{N_I}^{0,+}\}}],$$

which is the partition function of a coupled wetting model, with an additional ‘‘in-between area’’ penalization. Note that $(Q_N^{\beta,\delta,\gamma})_{N \geq 1}$ is super-multiplicative, therefore we can define the free energy of this model:

$$(5.12) \quad h_\beta(\delta, \gamma) := \lim_{N \rightarrow \infty} \frac{1}{N} \log Q_N^{\beta,\delta,\gamma},$$

Notice that $h_\beta(\delta, \gamma) \leq \delta$ (in particular h_β is finite), h_β is nonincreasing in γ and nondecreasing in δ , and it is continuous.

Recollecting (5.9), we conclude that $\tilde{f}(\beta, \delta)$ is the only positive solution (if it exists) in γ of

$$(5.13) \quad \log \Gamma_\beta - \gamma + h_\beta(\delta, \gamma) = 0,$$

and $\tilde{f}(\beta, \delta) = 0$ otherwise (this equation has at most one solution because $\gamma \mapsto -\gamma + h_\beta(\delta, \gamma)$ is decreasing).

Step 4: Characterizing the critical curve. Let $(\beta, \delta) \in \bar{\mathcal{C}} \cap \bar{\mathcal{E}}$ be a point of the critical curve: then $\tilde{f}(\beta, \delta) = 0$ (because \tilde{f} is continuous) and $\log \Gamma_\beta + h_\beta(\delta, 0) = 0$ (because (5.13) is continuous in $\gamma \searrow 0$). In particular, there is no area constraint in $Q_N^{\beta,\delta,0}$, and because S and I are independent we can uncouple them and write

$$(5.14) \quad Q_N^{\beta,\delta,0} = Z_{\text{wet},N_S}^{\beta,\delta} \times Z_{\text{wet},N_I}^{\beta,\delta},$$

where $Z_{\text{wet},N}^{\beta,\delta}$, $N \geq 1$ is the partition function of a wetting model (see (A.1)). By applying $\frac{1}{N} \log$ to (5.14), we notice that $h_\beta(\delta, 0)$ matches exactly the free energy of the wetting model: $h_\beta(\delta, 0) = h_\beta(\delta)$ (see (2.9)).

The asymptotic behavior of $Z_{\text{wet},N}^{\beta,\delta}$ is already well known—see Proposition A.1. Thus, we can finally conclude the proof of Theorem 2.2. Recall that $(\beta, \delta) \in \bar{\mathcal{C}} \cap \bar{\mathcal{E}}$ implies $\log \Gamma_\beta + h_\beta(\delta) = 0$, where Γ_β is decreasing in β , and β_c is the only solution of $\Gamma_\beta = 1$. Therefore:

- if $\beta < \beta_c$, then $\log \Gamma_\beta + h_\beta(\delta) \geq \log \Gamma_\beta > 0$ so $(\beta, \delta) \notin \bar{\mathcal{C}} \cap \bar{\mathcal{E}}$,
- if $\beta = \beta_c$, then $(\beta, \delta) \in \bar{\mathcal{C}} \cap \bar{\mathcal{E}}$ if and only if $h_\beta(\delta) = 0$, that is $\delta \leq \tilde{\delta}_c(\beta_c)$,
- if $\beta > \beta_c$, then there is a unique solution in $\delta \geq \tilde{\delta}_c(\beta)$ to $h_\beta(\delta) = -\log \Gamma_\beta$, which we note $\delta_c(\beta)$.

Moreover, recall that the left-hand side of (5.13) is decreasing and continuous in γ , so $\tilde{f}(\beta, \delta) = 0$ if and only if $\log \Gamma_\beta + h_\beta(\delta) \leq 0$, that is $\beta \geq \beta_c$ and $\delta \leq \delta_c(\beta)$; which fully characterizes \mathcal{C} . Finally, the analytic expression of $\delta_c(\beta)$ follows directly from (2.11) and (A.4) by solving a quadratic equation; details are left to the reader.

5.2. Proof of Proposition 3.1. This is very similar to Theorem 2.2. The single-bead partition $Z_{L,\beta,\delta}^{\text{bead}}$ function is already constrained to return to 0, hence it is super-multiplicative (and f^{bead} is well-posed) and we do not need to replicate Step 1. The random walk representation is already laid out in Step 2 of Section 5.1—notice that the main difference with (5.8) is the constraint $\{S > I\}$ (recall (4.5)), in particular the random walk S cannot touch the wall. Thereby the generating function of $\tilde{Z}_{L,\beta,\delta}^{\text{bead}}$ (defined in (4.7)) can be written (similarly to (5.10)) as

$$(5.15) \quad \sum_{L \geq 1} \tilde{Z}_{L,\beta,\delta}^{\text{bead}} e^{-\gamma L} = \sum_{N \geq 1} (\Gamma_\beta e^{-\gamma})^{2N} R_N^{\beta,\delta,\gamma},$$

we define

$$(5.16) \quad R_N^{\beta,\delta,\gamma} := \mathbf{E}_{\beta,0} [e^{-2\gamma(A_{N+1}(S) - A_N(I))} e^{\delta \sum_{k=1}^N \mathbf{1}_{\{I_k=0\}}} \mathbf{1}_{\{S > I\}} \mathbf{1}_{\{S \in B_{N+1}^{0,+}, I \in B_N^{0,+}\}}].$$

$(R_N^{\beta,\delta,\gamma})_{N \geq 1}$ is super-multiplicative, thereby we define its free energy

$$\kappa_\beta(\delta, \gamma) := \lim_{N \rightarrow \infty} \frac{1}{N} \log R_N^{\beta,\delta,\gamma},$$

and we note that $\tilde{f}^{\text{bead}}(\beta, \delta)$ is the only positive solution (if it exists) in γ of

$$2 \log \Gamma_\beta + 2\gamma + \kappa_\beta(\delta, \gamma) = 0,$$

and $\tilde{f}^{\text{bead}}(\beta, \delta) = 0$ otherwise.

Let $(\beta, \delta) \in \partial \mathcal{C}_{\text{bead}}$ be on the critical curve (which implies $\tilde{f}^{\text{bead}}(\beta, \delta) = 0$ and $2 \log \Gamma_\beta + \kappa_\beta(\delta, 0) = 0$). We notice $R_N^{\beta,\delta,\gamma} \leq Z_{\text{wet},N}^{\beta,\delta}$, so $\kappa_\beta(\delta, \gamma) \leq h_\beta(\delta)$, and we now compute a lower bound on $R_N^{\beta,\delta} := R_N^{\beta,\delta,0}$ to deduce $\kappa_\beta(\delta, 0) = h_\beta(\delta)$.

Let $\alpha > 0$ and recall (5.16) with $\gamma = 0$. Constraining I_k to remain below $\lfloor \alpha \sqrt{N} \rfloor$ for all $1 \leq k < N$, and constraining S to $S_1 = S_N = \lfloor \alpha \sqrt{N} \rfloor + 1$, and $S_k \geq \lfloor \alpha \sqrt{N} \rfloor + 1$ for all $2 \leq k < N$, we obtain the lower bound

$$\begin{aligned} R_N^{\beta,\delta} &\geq \mathbf{E}_{\beta,0} [e^{\delta \sum_{k=1}^N \mathbf{1}_{\{I_k=0\}}} \mathbf{1}_{\{I_k \leq \lfloor \alpha \sqrt{N} \rfloor, \forall 1 \leq k < N\}} \mathbf{1}_{\{I \in B_N^{0,+}\}}] \\ &\quad \times \mathbf{P}_{\beta,0}(S_1 = S_N = \lfloor \alpha \sqrt{N} \rfloor + 1, S_{N+1} = 0, S_k \geq \lfloor \alpha \sqrt{N} \rfloor + 1, \forall 2 \leq k < N) \\ &\geq \mathbf{E}_{\beta,0} [e^{\delta \sum_{k=1}^N \mathbf{1}_{\{I_k=0\}}} \mathbf{1}_{\{I_k \leq \lfloor \alpha \sqrt{N} \rfloor, \forall 1 \leq k < N\}} \mathbf{1}_{\{I \in B_N^{0,+}\}}] \left(\frac{e^{-\frac{\beta}{2} \lfloor \alpha \sqrt{N} \rfloor - \frac{\beta}{2}}}{c_\beta} \right)^2 \mathbf{P}_{\beta,0}(B_{N-1}^{0,+}), \end{aligned}$$

where we can estimate $\mathbf{P}_{\beta,0}(B_{N-1}^{0,+}) \asymp N^{-3/2}$ with (6.1) below. Moreover, we notice that

$$(x_k)_{k=1}^N \mapsto e^{\delta \sum_{k=1}^N \mathbf{1}_{\{x_k=0\}}}, \quad \text{and} \quad (x_k)_{k=1}^N \mapsto \mathbf{1}_{\{x_k \leq \lfloor \alpha \sqrt{N} \rfloor, \forall 1 \leq k < N\}},$$

are both bounded, nonincreasing functions on $B_N^{0,+}$, hence we can apply an FKG inequality (see Proposition B.1) to claim

$$(5.17) \quad \begin{aligned} \mathbf{E}_{\beta,0} & \left[e^{\delta \sum_{k=1}^N \mathbf{1}_{\{x_k=0\}}} \mathbf{1}_{\{I_k \leq \lfloor \alpha \sqrt{N} \rfloor, \forall 1 \leq k < N\}} \mathbf{1}_{\{I \in B_N^{0,+}\}} \right] \\ & \geq Z_{\text{wet},N}^{\beta,\delta} \times \mathbf{P}_{\beta,0}(I_k \leq \lfloor \alpha \sqrt{N} \rfloor, \forall 1 \leq k < N | I \in B_N^{0,+}), \end{aligned}$$

where we recall that $Z_{\text{wet},N}^{\beta,\delta}$, $N \geq 1$ is the partition function of a wetting model (see (A.1)), and the second factor in the right-hand side is bounded from below by some positive constant provided that α is large (see (6.2) below).

All this implies $\kappa_\beta(\delta, 0) \geq h_\beta(\delta)$, so $\kappa_\beta(\delta, 0) = h_\beta(\delta)$. Therefore, we conclude the proof of Proposition 3.1 similarly to Theorem 2.2.

6. Proofs of technical results. In this section we prove Lemmas 4.5 and 4.10, then the much more involved Proposition 4.3.

6.1. *Proof of Lemma 4.5.* Before proving Lemma 4.5, we provide a useful estimate on nonnegative random walk bridges proven in [3]. Recall the definition of $B_n^{y,+}$ in (2.10), and that $\mathbf{P}_{\beta,x}$ is the law of a random walk with increments distributed as \mathbf{P}_β (see (2.8)) starting from $x \geq 0$. One has

$$(6.1) \quad \mathbf{P}_{\beta,0}(B_n^{x,+}) = \mathbf{P}_{\beta,x}(B_n^{0,+}) \asymp \frac{\max(x, 1)}{n^{3/2}},$$

uniformly in $n \geq 1$ and $0 \leq x \leq C_1 \sqrt{n}$ for any $C_1 > 0$. The first identity is obtained by reversing the walk in time (notice that \mathbf{P}_β is symmetric), and the asymptotic behavior is derived in [3], (4.3).

It is also proven in [3], Corollary 2.5, that a properly rescaled centered random walk with finite variance conditioned to remain nonnegative and to finish in 0 converges (in distribution) to a Brownian excursion. The maximum being a continuous function on $\mathcal{C}([0, 1], \mathbb{R})$, we thereby deduce that for any $\eta > 0$, there exist $C_A > 0$ and $N_0 \in \mathbb{N}$ such that

$$(6.2) \quad \mathbf{P}_{\beta,0} \left(\max_{1 \leq k \leq N} (X_k) > C_A \sqrt{N} | X \in B_N^{0,+} \right) \leq \eta,$$

for all $N \geq N_0$.

PROOF OF LEMMA 4.5. Notice that $\max_{1 \leq k \leq N} (X_k) \leq C_A \sqrt{N}$ implies that $A_N(X) \leq C_A N^{3/2}$. Thus, we have the lower bound

$$(6.3) \quad \begin{aligned} & \mathbf{P}_{\beta,0}(\forall 1 \leq k < N, X_k \leq \tilde{f}_I(k); A_N(X) \leq C_A N^{3/2} | X \in B_N^{0,+}) \\ & \geq 1 - \mathbf{P}_{\beta,0} \left(\exists 1 \leq k < N, X_k > \tilde{f}_I(k); \max_{1 \leq k \leq N} (X_k) \leq C_A \sqrt{N} | X \in B_N^{0,+} \right) \\ & \quad - \mathbf{P}_{\beta,0} \left(\max_{1 \leq k \leq N} (X_k) > C_A \sqrt{N} | X \in B_N^{0,+} \right). \end{aligned}$$

Recalling (6.2), the last term in (6.3) is not larger than some $\eta > 0$ arbitrarily small. Regarding the other term, we write

$$(6.4) \quad \begin{aligned} & \mathbf{P}_{\beta,0} \left(\exists 1 \leq k < N, X_k > \tilde{f}_I(k); \max_{1 \leq k \leq N} (X_k) \leq C_A \sqrt{N} | X \in B_N^{0,+} \right) \\ & \leq 2 \sum_{k=1}^{\lceil N/2 \rceil} \mathbf{P}_{\beta,0}(\tilde{f}_I(k) < X_k \leq C_A \sqrt{N} | X \in B_N^{0,+}), \end{aligned}$$

where we reversed in time all terms with index larger than $\lceil N/2 \rceil$. For any $k \leq \lceil N/2 \rceil$, we partition the k th term over possible values of X_k , then we apply Markov property at time k to write

$$(6.5) \quad \mathbf{P}_{\beta,0}(\tilde{f}_I(k) < X_k \leq C_A \sqrt{N} | X \in B_N^{0,+}) = \sum_{z=\tilde{f}_I(k)+1}^{C_A \sqrt{N}} \mathbf{P}_{\beta,0}(B_k^{z,+}) \frac{\mathbf{P}_{\beta,z}(B_{N-k}^{0,+})}{\mathbf{P}_{\beta,0}(B_N^{0,+})}.$$

Recalling (6.1), we can bound $\mathbf{P}_{\beta,z}(B_{N-k}^{0,+})/\mathbf{P}_{\beta,0}(B_N^{0,+})$ from above by $C_2 \times z$ with $C_2 > 0$ a constant uniform in $z \leq C_A \sqrt{N}$ (recall that $N - k \geq \frac{N}{2}$). Hence, (6.5) becomes

$$\mathbf{P}_{\beta,0}(\tilde{f}_I(k) < X_k \leq C_A \sqrt{N} | X \in B_N^{0,+}) \leq C_2 \mathbf{E}_{\beta,0}[X_k \mathbf{1}_{\{\tilde{f}_I(k) < X_k \leq C_A \sqrt{N}\}} \mathbf{1}_{\{X_i \geq 0, \forall i \leq k\}}].$$

Recall the definition of \tilde{f}_I in (4.11). Noticing that $\mathbf{E}_{\beta,0}[e^{\lambda X_k}] = e^{k\mathcal{L}(\lambda)}$ and $\mathbf{E}_{\beta,0}[X_k e^{\lambda X_k}] = \partial_\lambda \mathbf{E}_{\beta,0}[e^{\lambda X_k}] = k\mathcal{L}'(\lambda)e^{k\mathcal{L}(\lambda)}$ for any $0 < \lambda < \beta/2$, and choosing λ sufficiently small so that $\mathcal{L}(\lambda) < \lambda\gamma$ (recall $\gamma > 0$ and $\mathcal{L}(\lambda) \sim \lambda^2 \sigma_\beta^2/2$ as $\lambda \searrow 0$), we obtain with Markov's inequality

$$\begin{aligned} \mathbf{P}_{\beta,0}(\tilde{f}_I(k) < X_k \leq C_A \sqrt{N} | X \in B_N^{0,+}) &\leq C_2 \mathbf{E}_{\beta,0}[X_k e^{\lambda(X_k - \gamma k - K_I)}] \\ &\leq C_3 k e^{-\lambda K_I} e^{-k(\lambda\gamma - \mathcal{L}(\lambda))}. \end{aligned}$$

The right-hand side being summable in k , there exists $C_4 > 0$ such that

$$(6.6) \quad \sum_{k=1}^{\lceil N/2 \rceil} \mathbf{P}_{\beta,0}(\tilde{f}_I(k) < X_k \leq C_A \sqrt{N} | X \in B_N^{0,+}) \leq C_4 e^{-\lambda K_I},$$

uniformly in $N \geq N_0$. Assuming K_I is sufficiently large, this term is smaller than any fixed $\eta > 0$. Recollecting (6.3), (6.2), (6.4) and (6.6) and fixing $\eta = \varepsilon/3$, this concludes the proof. \square

6.2. *Proof of Lemma 4.10.* Let us define for any $x, m \in \mathbb{N}_0$ some $\alpha > 0$,

$$(6.7) \quad D_{x,m} := \mathbf{E}_{\beta,x}[e^{-\alpha A_m(I)} | I \in B_m^+].$$

Let us prove that $x \mapsto D_{x,m}$ is nonincreasing for any $\alpha > 0, m \in \mathbb{N}_0$ by induction on m . Notice that the proof also holds when conditioning by $B_m^{0,+}$ instead of B_m^+ without further changes (this is required in the proof of Claim 1 in Section 4.4).

PROOF OF LEMMA 4.10. When $m = 0, D_{x,0} = e^{-\alpha x}$ for all $x \geq 0$, which is nonincreasing. Let $m \in \mathbb{N}$ and assume $x \mapsto D_{x,m-1}$ is nonincreasing. For any $x \geq 0$ and $\alpha > 0$, one has by Markov's property

$$(6.8) \quad \begin{aligned} \mathbf{E}_{\beta,x}[e^{-\alpha A_m(I)} | B_m^+] &= \sum_{y \geq 0} \mathbf{E}_{\beta,x}[e^{-\alpha x} e^{-\alpha \sum_{k=1}^m I_k} \mathbf{1}_{\{I_1=y\}} | B_m^+] \\ &= e^{-\alpha x} \sum_{y \geq 0} \mathbf{E}_{\beta,y}[e^{-\alpha A_{m-1}(I)} | B_{m-1}^+] R_x(y), \end{aligned}$$

where

$$(6.9) \quad R_x(y) := \frac{\mathbf{P}_{\beta,x}(I_1 = y) \mathbf{P}_{\beta,y}(B_{m-1}^+)}{\mathbf{P}_{\beta,x}(B_m^+)} = \mathbf{P}_{\beta,x}(I_1 = y | B_m^+).$$

For any $x, y \geq 0$, let $\bar{R}_x(y) := \sum_{t \geq y} R_x(t)$, so (6.8) becomes

$$\begin{aligned}
 & e^{\alpha x} \mathbf{E}_{\beta,x}[e^{-\alpha A_m(I)} | B_m^+] \\
 &= \sum_{y \geq 0} \mathbf{E}_{\beta,y}[e^{-\alpha A_{m-1}(I)} | B_{m-1}^+](\bar{R}_x(y) - \bar{R}_x(y+1)) \\
 (6.10) \quad &= \bar{R}_x(0) \mathbf{E}_{\beta,0}[e^{-\alpha A_{m-1}(I)} | B_{m-1}^+] \\
 &+ \sum_{y \geq 1} \bar{R}_x(y) (\mathbf{E}_{\beta,y}[e^{-\alpha A_{m-1}(I)} | B_{m-1}^+] - \mathbf{E}_{\beta,y-1}[e^{-\alpha A_{m-1}(I)} | B_{m-1}^+]).
 \end{aligned}$$

Recall that we assumed that $y \mapsto D_{y,m-1}$ is nonincreasing, so

$$\mathbf{E}_{\beta,y}[e^{-\alpha A_{m-1}(I)} | B_{m-1}^+] - \mathbf{E}_{\beta,y-1}[e^{-\alpha A_{m-1}(I)} | B_{m-1}^+] \leq 0,$$

for all $y \geq 1$.

Moreover, we claim that $\bar{R}_{x+1}(y) \geq \bar{R}_x(y)$ for all $x, y, m \geq 0$. To prove this, we first observe that for all $x, m \in \mathbb{N}_0$, and for any trajectory $(s_k)_{k=1}^m \in \mathbb{Z}^m$,

$$(6.11) \quad \mathbf{P}_{\beta,x+1}((I_k)_{k=1}^m = (s_k)_{k=1}^m) = e^{\pm \frac{\beta}{2}} \mathbf{P}_{\beta,x}((I_k)_{k=1}^m = (s_k)_{k=1}^m),$$

where the sign \pm is $+$ if $s_1 > x$ and $-$ otherwise; and then we distinguish the cases $y \geq x + 1$ (where the proof is instantaneous), and $y \leq x + 1$. For the latter case, it is obtained by induction on $0 \leq y \leq x + 1$ for a fixed $x \in \mathbb{N}_0$. To be more specific, when $y = 0$, one has $\bar{R}_x(0) = 1$ for all $x \geq 0$. Now assume that $\bar{R}_{x+1}(y) \geq \bar{R}_x(y)$ holds for some $0 \leq y \leq x$. Thus,

$$\begin{aligned}
 (6.12) \quad & \bar{R}_x(y+1) = \bar{R}_x(y) - \mathbf{P}_{\beta,x}(I_1 = y | B_m^+) \\
 & \leq \bar{R}_{x+1}(y) - \frac{\mathbf{P}_{\beta,x}(I_1 = y, I \in B_m^+)}{\mathbf{P}_{\beta,x}(B_m^+)}.
 \end{aligned}$$

Moreover, (6.11) ensures us on the one hand that $\mathbf{P}_{\beta,x}(B_m^+) \leq e^{\beta/2} \mathbf{P}_{x+1}(B_m^+)$, and on the other hand that $\mathbf{P}_{\beta,x}(I_1 = y, I \in B_m^+) = e^{\beta/2} \mathbf{P}_{\beta,x+1}(I_1 = y, I \in B_m^+)$ because $y \leq x$. Therefore, (6.12) becomes $\bar{R}_x(y+1) \leq \bar{R}_{x+1}(y) - \mathbf{P}_{\beta,x+1}(I_1 = y | B_m^+)$, which concludes the proof that $\bar{R}_{x+1}(y) \geq \bar{R}_x(y)$.

Noticing also that $\bar{R}_x(0) = \bar{R}_{x+1}(0) = 1$ for all $x \geq 0$, we obtain the lower bound

$$\begin{aligned}
 & e^{\alpha x} \mathbf{E}_{\beta,x}[e^{-\alpha A_m(I)} | B_m^+] \\
 & \geq \bar{R}_{x+1}(0) \mathbf{E}_{\beta,0}[e^{-\alpha A_{m-1}(I)} | B_{m-1}^+] \\
 & + \sum_{y \geq 1} \bar{R}_{x+1}(y) (\mathbf{E}_{\beta,y}[e^{-\alpha A_{m-1}(I)} | B_{m-1}^+] - \mathbf{E}_{\beta,y-1}[e^{-\alpha A_{m-1}(I)} | B_{m-1}^+]),
 \end{aligned}$$

and the identity (6.10) finally yields

$$e^{\alpha x} \mathbf{E}_{\beta,x}[e^{-\alpha A_m(I)} | B_m^+] \geq e^{\alpha(x+1)} \mathbf{E}_{\beta,x+1}[e^{-\alpha A_m(I)} | B_m^+],$$

which concludes the induction. \square

6.3. *Proof of Proposition 4.3.* As mentioned in Remark 4.4(iii), an equivalent of the probability that such random walk satisfies $(A_N, X_N) = (qN^2, pN)$ was first rigorously derived in [7]. The present proposition is an improvement of [4], Proposition 2.5, where a polynomial lower bound is displayed for the probability that an N -step walk (of law $\mathbf{P}_{\beta,0}$) remains positive and satisfies $(A_N, X_N) = (qN^2, 0)$. To that aim, we recall some tools introduced in [7] and used later on in [4]. We keep in mind that all the upcoming claims are proven

in [4]. Recall that $\mathcal{L}(h)$ is the logarithmic moment generating function of the increments of the random walk X ; see (3.9). To lighten upcoming formulae, let us define for any $N \in \mathbb{N}$,

$$(6.13) \quad \Lambda_N := (A_N(X)/N, X_N),$$

(notice that the area is normalized by n in this definition), as well as the parallelograms

$$(6.14) \quad \mathcal{D}_{\beta,N} := \{(h_0, h_1) \in \mathbb{R}^2; |h_1| < \beta/2, |(1 - 1/N)h_0 + h_1| < \beta/2\}.$$

For any $\mathbf{h} \in \mathcal{D}_{\beta,N}$, we define the tilted law:

$$(6.15) \quad \frac{d\mathbf{P}_{N,\mathbf{h}}}{d\mathbf{P}_{\beta,0}}(X) = e^{\mathbf{h} \cdot \Lambda_N - \mathcal{L}_{\Lambda_N}(\mathbf{h})}, \quad \mathcal{L}_{\Lambda_N}(\mathbf{h}) := \log \mathbf{E}_{\beta,0}[e^{\mathbf{h} \cdot \Lambda_N}],$$

where $\mathbf{h} \cdot \Lambda_N := h_0 A_N/N + h_1 X_N$ denotes the scalar product. Note that

$$(6.16) \quad \mathbf{h} \cdot \Lambda_N = \sum_{k=1}^N \left(\left(1 - \frac{k}{N}\right) h_0 + h_1 \right) (X_k - X_{k-1}),$$

so this change of measure is equivalent to tilt all increments of X independently, with an intensity depending on k . For any $q, p \in \mathbb{R}$ and $N \in \mathbb{N}$, let $\mathbf{h}_N^{q,p} = (h_{N,0}^{q,p}, h_{N,1}^{q,p})$ be the unique solution in $\mathbf{h} \in \mathcal{D}_{\beta,N}$ of the equation

$$(6.17) \quad \mathbf{E}_{N,\mathbf{h}} \left[\frac{1}{N} \Lambda_N \right] = \nabla \left[\frac{1}{N} \mathcal{L}_{\Lambda_N} \right] (\mathbf{h}) = (q, p),$$

where it is proven in [4], Lem. 5.4, that $\nabla[\frac{1}{N} \mathcal{L}_{\Lambda_N}]$ is a C^1 diffeomorphism from $\mathcal{D}_{\beta,N}$ to \mathbb{R}^2 . Notice that

$$(6.18) \quad \mathbf{P}_{\beta,0}(\Lambda_N = (Nq, Np)) = \mathbf{P}_{N,\mathbf{h}_N^{q,p}}(\Lambda_N = (Nq, Np)) e^{-N\mathbf{h}_N^{q,p} \cdot (q,p) + \mathcal{L}_{\Lambda_N}(\mathbf{h}_N^{q,p})}.$$

Moreover, [4], Proposition 6.1, gives a uniform local central limit theorem for $\mathbf{P}_{N,\mathbf{h}_N^{q,p}}$: for any $q_1 < q_2, p_1 < p_2, t_1 < t_2$ and $s_1 < s_2$, there are constants $C_5, C'_5 > 0$ and $N_0 \in \mathbb{N}$ such that for any $q \in [q_1, q_2], p \in [p_1, p_2], t \in [t_1, t_2], s \in [s_1, s_2]$ and $N \geq N_0$,

$$(6.19) \quad \frac{C_5}{N^2} \leq \mathbf{P}_{N,\mathbf{h}_N^{q,p}}(\Lambda_N = (qN + t\sqrt{N}, pN + s\sqrt{N})) \leq \frac{C'_5}{N^2}.$$

Notice that the authors of [4] only state this for $q \in [q_1, q_2]$ and $p = 0$, but their proof can be extended to $p \in [p_1, p_2]$ without further difficulty since the local limit theorem and all uniformity arguments for q hold similarly for $p \neq 0$.

The asymptotic of $\mathbf{h}_N^{q,p}$ for large N can be described sharply. For any $\mathbf{h} \in \mathcal{D}_{\beta}$, Recall that we defined $\mathcal{L}_{\Lambda}(\mathbf{h}) := \int_0^1 \mathcal{L}(h_0x + h_1) dx$ for all $\mathbf{h} \in \mathcal{D}_{\beta}$ (see (3.10)), and let $\tilde{\mathbf{h}}^{q,p} = (\tilde{h}_0^{q,p}, \tilde{h}_1^{q,p})$ be the unique solution in $\mathbf{h} \in \mathcal{D}_{\beta}$ of the equation $\nabla \mathcal{L}_{\Lambda}(\mathbf{h}) = (q, p)$, where

$$(6.20) \quad \begin{aligned} \nabla \mathcal{L}_{\Lambda}(\mathbf{h}) &= (\partial_{h_0} \mathcal{L}_{\Lambda}, \partial_{h_1} \mathcal{L}_{\Lambda})(\mathbf{h}) \\ &= \left(\int_0^1 x \mathcal{L}'(xh_0 + h_1) dx, \int_0^1 \mathcal{L}'(xh_0 + h_1) dx \right), \end{aligned}$$

is a C^1 diffeomorphism from \mathcal{D}_{β} to \mathbb{R}^2 (see [4], Lem. 5.3). If $h_0 \neq 0$, an integration by parts yields

$$(6.21) \quad \nabla \mathcal{L}_{\Lambda}(\mathbf{h}) = \frac{1}{h_0} (\mathcal{L}(h_0 + h_1) - \mathcal{L}_{\Lambda}(\mathbf{h}), \mathcal{L}(h_0 + h_1) - \mathcal{L}(h_1)),$$

and if $h_0 = 0, \nabla \mathcal{L}_{\Lambda}(\mathbf{h}) = (\mathcal{L}'(h_1)/2, \mathcal{L}'(h_1)).$

For any $q_1 < q_2$ and $p_1 < p_2 \in \mathbb{R}$, [4], Proposition 2.3, gives constants $C_6, C'_6 > 0$ and N_0 , such that for any $q \in [q_1, q_2]$, $p \in [p_1, p_2]$ and $N \geq N_0$, one has

$$(6.22) \quad \left| \left[\frac{1}{N} \mathcal{L}_{\Lambda_N}(\mathbf{h}_N^{q,p}) - \mathbf{h}_N^{q,p} \cdot (q, p) \right] - \left[\mathcal{L}_{\Lambda}(\tilde{\mathbf{h}}^{q,p}) - \tilde{\mathbf{h}}^{q,p} \cdot (q, p) \right] \right| \leq \frac{C_6}{N},$$

and

$$(6.23) \quad \|\mathbf{h}_N^{(q,p)} - \tilde{\mathbf{h}}^{q,p}\| \leq \frac{C'_6}{N}.$$

Notice that [4], Proposition 2.3, is only formulated for $p = 0$ and $q \in [q_1, q_2]$ with $q_1 > 0$, but all uniformity arguments also hold for $p \in [p_1, p_2]$ and any $[q_1, q_2] \subset \mathbb{R}$. These estimates, combined with (6.18) and (6.19), give the precise asymptotic of $\mathbf{P}_\beta(\Lambda_N = (Nq, Np))$ (up to constant factors) uniformly in $q \in [q_1, q_2]$, $p \in [p_1, p_2]$.

Before we start the proof, let us point out that $\tilde{h}_0^{q,p} > 0$ for all $q, p \in \mathbb{R}$ with $p \leq 2q - \varepsilon$. Indeed, one notices that $\tilde{h}_0^{q,p} = 0$ and (6.20) imply $p = 2q$. Because $\tilde{\mathbf{h}}$ is continuous, it has constant sign on each sets $\{p < 2q\}$ and $\{p > 2q\}$, and it is already stated in [4], Remark 5.5, that $\tilde{h}_0^{q,0} > 0$ for all $q > 0$.

Finally we also define the i.i.d. uniform tilt of the increments of X for any $|\delta| < \beta/2$:

$$(6.24) \quad \frac{d\tilde{\mathbf{P}}_\delta}{d\mathbf{P}_{\beta,0}}(X_1) = e^{\delta X_1 - \mathcal{L}(\delta)},$$

where we defined \mathcal{L} in (3.9).

PROOF OF PROPOSITION 4.3. Here is the strategy of the proof: recollecting (6.18), (6.19) and (6.22), we already have

$$(6.25) \quad \mathbf{P}_{\beta,0}(\Lambda_N = (qN, pN)) \asymp \frac{1}{N^2} e^{-N(\tilde{\mathbf{h}}^{q,p} \cdot (q,p) - \mathcal{L}_\Lambda(\tilde{\mathbf{h}}^{q,p}))},$$

(recall (0.2) for the definition of \asymp). Hence, the proof will be complete when we show that

$$\mathbf{P}_{\beta,0}(X_k \geq \tilde{f}(k), \forall 0 < k < N | \Lambda_N = (qN, pN))$$

is bounded from below by some positive constant uniform in $q \in [q_1, q_2]$, $p \in [p_1, p_2]$ and $N \in \mathbb{N}$.

For that purpose, we decompose a trajectory X with $X_0 = 0$, $X_N = pN$ and area $A_N(X) = qN^2$ into three parts: both ends of length $u_N := N^{1/3}$, and the middle part of length $N' := N - 2u_N$. To estimate the probability to go under the fixed curve $\tilde{f}(k)$, we study the middle part under the (nonuniform) tilt $\mathbf{P}_{N',\mathbf{h}}$, and we handle both ends with uniform tilts $\tilde{\mathbf{P}}_{\delta_1}, \tilde{\mathbf{P}}_{\delta_2}$. Then we take advantage of the fact that uniformly tilted walks remain positive (and even above certain affine curves) with positive probability, and we handle the middle part with aforementioned estimates.

For $i \in \{1, 2\}$, define

$$(6.26) \quad Q_{i,u_N} := \left([a_i u_N / 2, b_i u_N / 2 + 3K + K_S] \cap \left(\frac{1}{u_N} \mathbb{Z} \right) \right) \times \left([a_i(u_N - 1) + K + K_S, b_i(u_N - 1) + 3K + K_S] \cap \mathbb{Z} \right),$$

where $a_i < b_i$, $i \in \{1, 2\}$ and K are convenient positive constants which will be fixed below. We constrain the first bit of the walk (until index u_N) to grow by $X_{u_N} = x_1$ and have an area $A_{u_N} = A_1$, with $(A_1/u_N, x_1) \in Q_{1,u_N}$. Similarly, we reverse the third bit in time (from N to $N - u_N$), and we constrain it to grow by $X_{N-u_N} - pN = x_2$ and have an area $(A_N - A_{N-u_N}) - pNu_N = A_2$, with $(A_2/u_N, x_2) \in Q_{2,u_N}$.

Therefore we have the decomposition

$$\begin{aligned}
 & \mathbf{P}_{\beta,0}(X_k \geq \tilde{f}(k), \forall 0 < k < N; \Lambda_N = (qN, pN)) \\
 & \geq \sum_{\substack{(A_1/u_N, x_1) \in Q_{1,u_N}, \\ (A_2/u_N, x_2) \in Q_{2,u_N}}} \mathbf{P}_{\beta,0}(X_k \geq \tilde{f}(k), \forall 0 < k < u_N; \Lambda_{u_N} = (A_1/u_N, x_1)) \\
 (6.27) \quad & \times \mathbf{P}_{\beta,0}(X_k \geq \tilde{f}(k + u_N) - x_1, \forall 0 < k < N'; \Lambda_{N'} = (q'N', p'N')) \\
 & \times \mathbf{P}_{\beta,0}(X_k \geq \tilde{f}(N - k) - pN, \forall 0 < k < u_N; \Lambda_{u_N} = (A_2/u_N, x_2)),
 \end{aligned}$$

where $N' := N - 2u_N$, and we define (q', p') such that $q'N'^2 = qN^2 - A_1 - A_2 - x_1N' - pNu_N$ and $p'N' = pN + x_2 - x_1$. Notice that with our choice of $A_1, A_2 = O(u_N^2), x_1, x_2 = O(u_N)$, with $u_N = N^{1/3}$, and because $q \in [q_1, q_2], p \in [p_1, p_2]$, then for N sufficiently large q' and p' are contained in some compact sets $[q'_1, q'_2]$ and $[p'_1, p'_2]$. Moreover, we have the following estimates:

$$\begin{aligned}
 (6.28) \quad & q - q' = (p' - 4q') \frac{u_N}{N} + \frac{x_1}{N} + o(1/N), \\
 & p - p' = -2p' \frac{u_N}{N} + \frac{x_1 - x_2}{N}.
 \end{aligned}$$

As mentioned above, we tilt uniformly the first and last factors in the right-hand side of (6.27) with respective parameters δ_1, δ_2 (which will be explicitly fixed later),

$$\begin{aligned}
 & \mathbf{P}_{\beta,0}(X_k \geq \tilde{f}(k), \forall 0 < k < u_N; \Lambda_{u_N} = (A_1/u_N, x_1)) \\
 & = e^{-\delta_1 x_1 + u_N \mathcal{L}(\delta_1)} \tilde{\mathbf{P}}_{\delta_1}(X_k \geq \tilde{f}(k), \forall 0 < k < u_N; \Lambda_{u_N} = (A_1/u_N, x_1)); \\
 (6.29) \quad & \mathbf{P}_{\beta,0}(X_k \geq \tilde{f}(N - k) - pN, \forall 0 < k < u_N; \Lambda_{u_N} = (A_2/u_N, x_2)) \\
 & = e^{-\delta_2 x_2 + u_N \mathcal{L}(\delta_2)} \tilde{\mathbf{P}}_{\delta_2}(X_k \geq \tilde{f}(N - k) - pN, \forall 0 < k < u_N; \Lambda_{u_N} = (A_2/u_N, x_2)).
 \end{aligned}$$

The second factor in the right-hand side of (6.27) is bounded from below by the following lemma.

LEMMA 6.1. *Fix some $\varepsilon > 0$ and $q'_1 < q'_2, p'_1 < p'_2$ in \mathbb{R} . Then there are constants $c, C > 0$ and $N_0 \in \mathbb{N}$ such that for any $N' \geq N_0, q' \in [q'_1, q'_2]$ and $p' \in [p'_1, p'_2]$ satisfying $p' \leq 2q' - \varepsilon$, one has*

$$\begin{aligned}
 (6.30) \quad & \mathbf{P}_{\beta,0}(X_k \geq \tilde{f}(k + u_N) - x_1, \forall 0 < k < N'; \Lambda_{N'} = (q'N', p'N')) \\
 & \geq e^{-N' \mathbf{h}_{N'}^{q',p'} \cdot (q', p') + \mathcal{L}_{\Lambda_{N'}}(\mathbf{h}_{N'}^{q',p'})} (\mathbf{P}_{N', \mathbf{h}_{N'}^{q',p'}}(\Lambda_{N'} = (q'N', p'N')) - Ce^{-cu_N}).
 \end{aligned}$$

This lemma is proven afterwards. Writing (6.27) with (6.29) and (6.30), we have

$$\begin{aligned}
 & \mathbf{P}_{\beta,0}(X_k \geq \tilde{f}(k), \forall 0 < k < N; \Lambda_N = (qN, pN)) \\
 & \geq \sum_{\substack{(A_1/u_N, x_1) \in Q_{1,u_N}, \\ (A_2/u_N, x_2) \in Q_{2,u_N}}} e^{-\delta_1 x_1 + u_N \mathcal{L}(\delta_1)} e^{-\delta_2 x_2 + u_N \mathcal{L}(\delta_2)} e^{-N' \mathbf{h}_{N'}^{q',p'} \cdot (q', p') + \mathcal{L}_{\Lambda_{N'}}(\mathbf{h}_{N'}^{q',p'})} \\
 (6.31) \quad & \times (\mathbf{P}_{N', \mathbf{h}_{N'}^{q',p'}}(\Lambda_{N'} = (q'N', p'N')) - Ce^{-cu_N}) \\
 & \times \tilde{\mathbf{P}}_{\delta_1}(X_k \geq \tilde{f}(k), \forall 0 < k < u_N; \Lambda_{u_N} = (A_1/u_N, x_1)) \\
 & \times \tilde{\mathbf{P}}_{\delta_2}(X_k \geq \tilde{f}(N - k) - pN, \forall 0 < k < u_N; \Lambda_{u_N} = (A_2/u_N, x_2)).
 \end{aligned}$$

Then we divide by $\mathbf{P}_{\beta,0}(\Lambda_N = (qN, pN))$, which we express with the tilt $\mathbf{P}_{N, \mathbf{h}_N^{q', p'}}$ (rather than $\mathbf{P}_{N, \mathbf{h}_N^{q, p}}$), so we have

$$\begin{aligned}
 & \mathbf{P}_{\beta,0}(X_k \geq \tilde{f}(k), \forall 0 < k < N | \Lambda_N = (qN, pN)) \\
 (6.32) \quad & \geq \sum_{\substack{(A_1/u_N, x_1) \in \mathcal{Q}_{1, u_N}, \\ (A_2/u_N, x_2) \in \mathcal{Q}_{2, u_N}}} e^{g(x_1, a_1, x_2, a_2)} \left(\frac{\mathbf{P}_{N', \mathbf{h}_{N'}^{q', p'}}(\Lambda_{N'} = (q'N', p'N')) - Ce^{-cu_N}}{\mathbf{P}_{N, \mathbf{h}_N^{q', p'}}(\Lambda_N = (qN, pN))} \right) \\
 & \times \tilde{\mathbf{P}}_{\delta_1}(X_k \geq \tilde{f}(k), \forall 0 < k < u_N; \Lambda_{u_N} = (A_1/u_N, x_1)) \\
 & \times \tilde{\mathbf{P}}_{\delta_2}(X_k \geq \tilde{f}(N - k) - pN, \forall 0 < k < u_N; \Lambda_{u_N} = (A_2/u_N, x_2)),
 \end{aligned}$$

where we define

$$\begin{aligned}
 (6.33) \quad g(x_1, a_1, x_2, a_2) := & -\delta_1 x_1 + u_N \mathcal{L}(\delta_1) - \delta_2 x_2 + u_N \mathcal{L}(\delta_2) - N' \mathbf{h}_{N'}^{q', p'} \cdot (q', p') \\
 & + \mathcal{L}_{\Lambda_{N'}}(\mathbf{h}_{N'}^{q', p'}) + N \mathbf{h}_N^{q', p'} \cdot (q, p) - \mathcal{L}_{\Lambda_N}(\mathbf{h}_N^{q', p'}).
 \end{aligned}$$

Let us handle the factor in parenthesis in (6.32) first. We can apply (6.19) to both probabilities (notice that $q - q', p - p'$ are of order $u_N/N \ll N^{-1/2}$, recall (6.28)) to bound it from below by $(C_7 \frac{N^2}{N^2} - C_8 N^2 e^{-cu_N}) \geq C_9 > 0$ for N sufficiently large, uniformly in q, p .

Now let us focus on $g(x_1, a_1, x_2, a_2)$, where we fix $\delta_1 = \tilde{h}_0^{q', p'} + \tilde{h}_1^{q', p'}$ and $\delta_2 = -\tilde{h}_1^{q', p'}$. Notice that they match respectively, the value of the exponential tilt applied on the first and last increments in the second piece of trajectory and in the limit $N \rightarrow \infty$ (recall (6.16), and the sign of δ_2 is inverted because we reversed in time the third bit of the trajectory). We introduce in (6.33) a term $N \mathbf{h}_N^{q', p'} \cdot (q', p')$ and we apply (6.22) for (q', p', N') and (q', p', N) to bound from below

$$\begin{aligned}
 (6.34) \quad g(x_1, a_1, x_2, a_2) \geq & -\delta_1 x_1 + u_N \mathcal{L}(\delta_1) - \delta_2 x_2 + u_N \mathcal{L}(\delta_2) \\
 & + N \mathbf{h}_N^{q', p'} \cdot (q - q', p - p') \\
 & - N' (\tilde{\mathbf{h}}^{q', p'} \cdot (q', p') - \mathcal{L}_\Lambda(\tilde{\mathbf{h}}^{q', p'})) \\
 & + N (\tilde{\mathbf{h}}^{q', p'} \cdot (q', p') - \mathcal{L}_\Lambda(\tilde{\mathbf{h}}^{q', p'})) - C_{10},
 \end{aligned}$$

for some uniform constant $C_{10} < \infty$. Recall $N' = N - 2u_N$, and apply (6.23) to the term $N \mathbf{h}_N^{q', p'} \cdot (q - q', p - p')$ to obtain

$$\begin{aligned}
 (6.35) \quad g(x_1, a_1, x_2, a_2) \geq & -\delta_1 x_1 + u_N \mathcal{L}(\delta_1) - \delta_2 x_2 + u_N \mathcal{L}(\delta_2) \\
 & + N \tilde{\mathbf{h}}^{q', p'} \cdot (q - q', p - p') \\
 & + 2u_N (\tilde{\mathbf{h}}^{q', p'} \cdot (q', p') - \mathcal{L}_\Lambda(\tilde{\mathbf{h}}^{q', p'})) - C_{11}.
 \end{aligned}$$

Recalling that $\delta_1 = \tilde{h}_0^{q', p'} + \tilde{h}_1^{q', p'}$ and $\delta_2 = -\tilde{h}_1^{q', p'}$, and also (6.28), we have

$$\begin{aligned}
 (6.36) \quad g(x_1, a_1, x_2, a_2) \geq & -(\tilde{h}_0^{q', p'} + \tilde{h}_1^{q', p'})x_1 + h_1^{q', p'} x_2 \\
 & + u_N (2 \tilde{\mathbf{h}}^{q', p'} \cdot (q', p') - 2 \mathcal{L}_\Lambda(\tilde{\mathbf{h}}^{q', p'})) \\
 & + \mathcal{L}(\tilde{h}_0^{q', p'} + \tilde{h}_1^{q', p'}) + \mathcal{L}(-\tilde{h}_1^{q', p'}) \\
 & + \tilde{h}_0^{q', p'} ((p' - 4q')u_N + x_1) \\
 & + \tilde{h}_1^{q', p'} (-2p'u_N + x_1 - x_2) - C_{12}.
 \end{aligned}$$

So,

$$(6.37) \quad \begin{aligned} g(x_1, a_1, x_2, a_2) &\geq u_N(-2\tilde{h}_0^{q',p'} q' + \tilde{h}_0^{q',p'} p' - 2\mathcal{L}_\Lambda(\tilde{h}^{q',p'})) \\ &\quad + \mathcal{L}(\tilde{h}_0^{q',p'} + \tilde{h}_1^{q',p'}) + \mathcal{L}(-\tilde{h}_1^{q',p'}) - C_{12}, \end{aligned}$$

and we conclude by recalling the definition of $\tilde{h}^{q',p'}$ and (6.21): the first term is zero, and $g(x_1, a_1, x_2, a_2)$ is bounded from below uniformly by some constant.

Therefore, we can bound (6.32) from below by

$$(6.38) \quad \begin{aligned} &\mathbf{P}_{\beta,0}(X_k \geq \tilde{f}(k), \forall 0 < k < N | \Lambda_N = (qN, pN)) \\ &\geq C_{13} \times \tilde{\mathbf{P}}_{\delta_1}(X_k \geq \tilde{f}(k), \forall 0 < k < u_N; \Lambda_{u_N} \in \mathcal{Q}_{1,u_N}) \\ &\quad \times \tilde{\mathbf{P}}_{\delta_2}(X_k \geq \tilde{f}(N - k) - pN, \forall 0 < k < u_N; \Lambda_{u_N} \in \mathcal{Q}_{2,u_N}), \end{aligned}$$

with $\delta_1 = \tilde{h}_0^{q',p'} + \tilde{h}_1^{q',p'}$, $\delta_2 = -\tilde{h}_1^{q',p'}$, and C_{13} is some uniform positive constant. The proof will be over once we show that those factors are uniformly bounded away from 0. We focus on the first factor, because we notice in the second one $\tilde{f}(N - k) - pN = Nf(\frac{k}{N}) - kp + K_S$, which matches $\tilde{f}(k)$ with an inverted slope $-p$ (recall we reversed in time that bit); hence it is handled by the exact same arguments.

Recall that $\nabla\mathcal{L}_\Lambda$ is a C^1 diffeomorphism, in particular \tilde{h} is Lipschitz on compact sets, and $(q - q')$, $(p - p')$ are of order $u_N/N = N^{-2/3}$ (recall (6.28)), so there exists some constant $C_{14} > 0$ such that for all $q \in [q_1, q_2]$, $p \in [p_1, p_2]$ and $N \in \mathbb{N}$,

$$(6.39) \quad |\delta_1 - \tilde{h}_0^{p,q} - \tilde{h}_1^{p,q}| \leq \frac{C_{14}u_N}{N}.$$

In particular for N large enough, δ_1 is contained in some compact set $[\delta, \bar{\delta}] \subset (-\beta/2, \beta/2)$ uniformly in q, p . As claimed earlier we have $\tilde{h}_0^{q,p} > 0$ under our assumptions—it is even bounded away from 0—so we also have $\tilde{h}_0^{q',p'} > 0$ for N large (recall (6.28) again and \tilde{h} is locally Lipschitz). Combined with (6.21) and with the strict convexity of \mathcal{L} , this implies

$$(6.40) \quad \mathcal{L}'(\tilde{h}_1^{q',p'}) < p' < \mathcal{L}'(\tilde{h}_0^{q',p'} + \tilde{h}_1^{q',p'}).$$

In particular there is some constant $C_{15} > 0$ such that $\mathcal{L}'(\delta_1) - p' \geq C_{15}$ and $p' + \mathcal{L}'(\delta_2) \geq C_{15}$ uniformly in $q \in [q_1, q_2]$, $p \in [p_1, p_2]$ and $N \in \mathbb{N}$ (recall that those functions are continuous, and $\mathcal{L}'(\tilde{h}_1^{q',p'}) = -\mathcal{L}'(-\tilde{h}_1^{q',p'})$).

Fix some $a, C > 0$. For any δ such that $|\delta| < \beta/2$, and any $0 < t < \beta/2 + \delta$, $k \in \mathbb{N}$, Markov’s inequality implies that

$$(6.41) \quad \tilde{\mathbf{P}}_\delta(X_k \leq ak - C) \leq \tilde{\mathbf{E}}_\delta[e^{t(ak-C)-tX_k}] = e^{-Ct} e^{k(\tilde{\mathcal{L}}_\delta(-t)+at)},$$

where $\tilde{\mathcal{L}}_\delta(-t) := \log \tilde{\mathbf{E}}_\delta[e^{-tX_1}] = \mathcal{L}(\delta - t) - \mathcal{L}(\delta)$ for any $t \in (-\beta/2 + \delta, \beta/2 + \delta)$. Choosing $a = \frac{3}{4}\mathcal{L}'(\delta_1) + \frac{1}{4}p'$ and writing a Taylor expansion of \mathcal{L} , there is some (uniform) $c > 0$ such that for t small,

$$(6.42) \quad \tilde{\mathcal{L}}_{\delta_1}(-t) + at \leq \frac{t}{4}(p' - \mathcal{L}'(\delta_1)) + ct^2.$$

Recall that $p' - \mathcal{L}'(\delta_1) < 0$, and it is even bounded away from 0 (uniformly). So for t sufficiently small, we have $\tilde{\mathcal{L}}_{\delta_1}(-t) + at \leq \frac{t}{8}(p' - \mathcal{L}'(\delta_1))$. Hence, there exists a $t_0 > 0$ such that

$$(6.43) \quad \begin{aligned} \tilde{\mathbf{P}}_{\delta_1}(\exists k \geq 1, X_k \leq ak - C) &\leq \sum_{k \geq 1} \tilde{\mathbf{P}}_{\delta_1}(X_k \leq ak - C) \\ &\leq e^{-Ct_0} \frac{e^{t_0(p' - \mathcal{L}'(\delta_1))/8}}{1 - e^{t_0(p' - \mathcal{L}'(\delta_1))/8}} \leq C_{16}e^{-Ct_0}, \end{aligned}$$

where we bounded the fraction by some constant uniform in (q', p') .

Similarly, we have

$$(6.44) \quad \tilde{\mathbf{P}}_\delta(X_k \geq bk + C) \leq \tilde{\mathbf{E}}_\delta[e^{wX_k - w(bk+C)}] = e^{-Cw} e^{k(\tilde{\mathcal{L}}_\delta(w) - bw)},$$

for some $b, C > 0$ fixed, and any $k \in \mathbb{N}$, δ such that $|\delta| < \beta/2$, and $0 < w < \beta/2 - \delta$. Because $\tilde{\mathcal{L}}_\delta(w) \sim \mathcal{L}'(\delta)w$ as $w \rightarrow 0$, we fix some $b > \sup\{\mathcal{L}'(\delta), \delta \in [\underline{\delta}, \bar{\delta}]\}$, and for some $w_0 > 0$ sufficiently small we have

$$(6.45) \quad \tilde{\mathbf{P}}_{\delta_1}(\exists k \geq 1, X_k \geq bk + C) \leq e^{-Cw_0} \frac{e^{\tilde{\mathcal{L}}_{\delta_1}(w_0) - bw_0}}{1 - e^{\tilde{\mathcal{L}}_{\delta_1}(w_0) - bw_0}} \leq C_{17}e^{-Cw_0},$$

where we bounded the fraction by some uniform constant (notice that we could pick $w_0 > 0$ such that $\tilde{\mathcal{L}}_\delta(w_0) - bw_0 < 0$ for all $\delta \in [\underline{\delta}, \bar{\delta}]$).

Finally, we estimate the first factor in (6.38) by constraining the trajectory X to have a first step $X_1 = 2K + K_S$, then to remain between the two curves $a(k - 1) + K + K_S$ and $b(k - 1) + 3K + K_S$, $k \geq 2$. Moreover for $\gamma > 0$ sufficiently small, we have

$$(6.46) \quad a - (p + \gamma) = \frac{3}{4}(\mathcal{L}'(\delta_1) - p') - \gamma + (p' - p) > 0,$$

(recall that $p' - p = O(N^{-2/3})$ and $\mathcal{L}'(\delta_1) - p'$ is uniformly bounded away from 0), and this γ can be chosen independently of K_S . This means that our constraint implies that X_k remains above $\tilde{f}(k) = \gamma k + kp + K_S$ for all $1 \leq k \leq u_N$. It also implies $\Lambda_{u_N} \in \mathcal{Q}_{1, u_N}$ by setting $a_1 = a = \frac{3}{4}\mathcal{L}'(\delta_1) + \frac{1}{4}p'$ and $b_1 = b$ in the definition of \mathcal{Q}_{1, u_N} . Recollecting (6.43) and (6.45), we finally obtain

$$(6.47) \quad \begin{aligned} &\tilde{\mathbf{P}}_{\delta_1}(X_k \geq \tilde{f}(k), \forall 0 < k < u_N; \Lambda_{u_N} \in \mathcal{Q}_{1, u_N}) \\ &\geq \tilde{\mathbf{P}}_{\delta_1}(X_1 = 2K + K_S; a(k - 1) + K + K_S \leq X_k \leq b(k - 1) + 3K + K_S, \forall k \geq 2) \\ &= \tilde{\mathbf{P}}_{\delta_1}(X_1 = 2K + K_S) \times \tilde{\mathbf{P}}_{\delta_1}(ak - K \leq X_k \leq bk + K, \forall k \geq 1) \\ &\geq \tilde{\mathbf{P}}_{\delta_1}(X_1 = 2K + K_S) \times (1 - C_{18}e^{-K \min(t_0, w_0)}) \geq C_{19} > 0. \end{aligned}$$

where we choose $K > 0$ such that the second factor in (6.47) is greater than $\frac{1}{2}$, then bound $\tilde{\mathbf{P}}_{\delta_1}(X_1 = 2K + K_S)$ uniformly over $\delta_1 \in [\underline{\delta}, \bar{\delta}]$. We can reproduce the same proof to handle the other factor in (6.38), by setting $a_2 = \frac{3}{4}\mathcal{L}'(\delta_2) - \frac{1}{4}p'$ and a convenient $b_2 > a_2$ in the definition of \mathcal{Q}_{2, u_N} . Therefore, (6.38) is bounded from below by some uniform positive constant, and this concludes the proof of the proposition (subject to Lemma 6.1). \square

Let us now prove Lemma 6.1. This result relies on estimates that are similar to those displayed at the end of the proof of Proposition 4.3, where we used Markov inequalities for $\tilde{\mathbf{P}}_\delta$. First we prove an upper bound on the moment generating function of the difference between the random walk with law $\mathbf{P}_{N', h^{q,p}}_{N'}$ and the straight line with slope p .

LEMMA 6.2. *There exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$, there are $c > 0$ and $N_0 \in \mathbb{N}$ such that*

$$(6.48) \quad \mathbf{E}_{N, h^{q,p}}[e^{-\lambda(X_k - pk)}] \leq e^{-ck},$$

uniformly in $0 \leq k \leq N$, $N \geq N_0$, $q \in [q_1, q_2]$, $p \in [p_1, p_2]$ satisfying $p \leq 2q - \varepsilon$.

PROOF. Under the law $\mathbf{P}_{N, \mathbf{h}_N^{q,p}}$, the increments of the random walk X are still independent (but no more identically distributed), so we define $h_N^i := (1 - i/N)h_{N,0}^{q,p} + h_{N,1}^{q,p}$, the tilt parameter for the increment $X_i - X_{i-1}$, $i \geq 1$. Then we have

$$\begin{aligned}
 \log \mathbf{E}_{N, \mathbf{h}_N^{q,p}} [e^{-\lambda(X_k - pk)}] &= \lambda pk + \sum_{i=1}^k (\mathcal{L}(-\lambda + h_N^i) - \mathcal{L}(h_N^i)) \\
 (6.49) \qquad \qquad \qquad &= -\lambda \sum_{i=1}^k \left(\frac{\mathcal{L}(h_N^i) - \mathcal{L}(-\lambda + h_N^i)}{\lambda} - p \right) \\
 &\leq -\lambda \sum_{i=1}^k (\mathcal{L}'(-\lambda + h_N^i) - p),
 \end{aligned}$$

where we used the convexity of \mathcal{L} , and the right-hand side is well defined for all $|\lambda| < \lambda_0$, for some $\lambda_0 > 0$ uniform in (q, p) (recall that $\mathbf{h}_N^{q,p}$ is contained in a compact subset of $\mathcal{D}_{\beta,n}$). Notice that h_N^i is affine, decreasing in i (recall that $p \leq 2q - \varepsilon$ implies $\tilde{h}_0^{q,p} > 0$), and recall that \mathcal{L}' is increasing. Therefore it suffices to prove the claim for $k = N$, that is to prove

$$(6.50) \qquad \qquad \qquad \frac{1}{N} \sum_{i=1}^N \mathcal{L}'(-\lambda + h_N^i) - p \geq c > 0,$$

and we conclude by noticing that if $(u_n)_{n \geq 1}$ is a decreasing sequence, then $(\frac{1}{n} \sum_{i=1}^n u_i)_{n \geq 1}$ is decreasing too, hence the claim holds for $k \leq N$.

Notice that the first term in the left-hand side of (6.50) is a Riemann sum, and we have

$$(6.51) \qquad \qquad \qquad \frac{1}{N} \sum_{i=1}^N \mathcal{L}'(-\lambda + h_N^i) \xrightarrow{N \rightarrow \infty} \int_0^1 \mathcal{L}'(x \tilde{h}_0^{q,p} + \tilde{h}_1^{q,p} - \lambda) dx.$$

We also claim that the convergence holds uniformly in $q \in [q_1, q_2]$, $p \in [p_1, p_2]$ —this follows from a Riemann-sum approximation of the right-hand side, from (6.23) and from the fact that \mathcal{L}' is locally Lipschitz. Moreover, we have $p = \int_0^1 \mathcal{L}'(x \tilde{h}_0^{q,p} + \tilde{h}_1^{q,p}) dx$ (recall (6.20)), so this concludes the proof of (6.50) for N sufficiently large and some $c > 0$ uniform in $q \in [q_1, q_2]$, $p \in [p_1, p_2]$. \square

PROOF OF LEMMA 6.1. Let us bound from above

$$\mathbf{P}_{\beta,0}(X_k < \tilde{f}(k + u_N) - x_1; \Lambda_{N'} = (q'N', p'N')),$$

for any $0 < k < N'$, where we recall $x_1 \in [a_1(u_N - 1) + K + K_S, b_1(u_N - 1) + 3K + K_S]$, $u_N = N^{1/3}$, $N' = N - 2u_N$ and a_1 satisfies (6.46) with $a = a_1$. We have

$$\begin{aligned}
 (6.52) \qquad \mathbf{P}_{\beta,0}(X_k < \tilde{f}(k + u_N) - x_1; \Lambda_{N'} = (q'N', p'N')) \\
 \leq \mathbf{P}_{\beta,0}(X_k < -\zeta u_N + pk + \gamma(k \wedge (N' - k)) - K + a_1; \Lambda_{N'} = (q'N', p'N')),
 \end{aligned}$$

where we define $\zeta := a_1 - p - \gamma$ (which is positive and bounded away from 0 uniformly in p). Recalling (6.15) and applying Markov’s inequality, we have

$$\begin{aligned}
 (6.53) \qquad \mathbf{P}_{\beta,0}(X_k < \tilde{f}(k + u_N) - x_1; \Lambda_{N'} = (q'N', p'N')) \\
 \leq e^{-N' \mathbf{h}_{N'}^{q',p'} \cdot (q', p') + \mathcal{L}_{\Lambda_{N'}}(\mathbf{h}_{N'}^{q',p'})} \\
 \times \mathbf{P}_{N', \mathbf{h}_{N'}^{q',p'}}(X_k < -\zeta u_N + pk + \gamma(k \wedge (N' - k)) - K + a_1; \Lambda_{N'} = (q'N', p'N')) \\
 \leq e^{-N' \mathbf{h}_{N'}^{q',p'} \cdot (q', p') + \mathcal{L}_{\Lambda_{N'}}(\mathbf{h}_{N'}^{q',p'})} \\
 \times e^{-\lambda \zeta u_N + \lambda \gamma(k \wedge (N' - k)) + \lambda(a_1 - K)} \mathbf{E}_{N', \mathbf{h}_{N'}^{q',p'}} [e^{-\lambda(X_k - p'k)}] e^{\lambda(p-p')k},
 \end{aligned}$$

for some $\lambda \in (0, \lambda_0)$. Applying Lemma 6.2 and the inequality $k \wedge (N' - k) \leq k$, we obtain

$$(6.54) \quad \begin{aligned} & \mathbf{P}_{\beta,0}(X_k < \tilde{f}(k + u_N) - x_1; \Lambda_{N'} = (q'N', p'N')) \\ & \leq e^{-N'h_{N'}^{q',p'} \cdot (q',p') + \mathcal{L}_{\Lambda_{N'}}(h_{N'}^{q',p'})} e^{-\lambda\zeta u_N + \lambda\gamma k + \lambda(a_1 - K)} e^{-ck} e^{\lambda(p-p')k}. \end{aligned}$$

Choosing $\gamma < c/\lambda$ (independently of K_S), and recalling that $p - p'$ goes uniformly to 0 (see (6.28)), there is $N_0 \in \mathbb{N}$ such that $\lambda\gamma - c + \lambda(p - p') \leq -C_{20} < 0$ uniformly in (q, p) and $N \geq N_0$. Therefore, we have

$$(6.55) \quad \begin{aligned} & \mathbf{P}_{\beta,0}(X_k \geq \tilde{f}(k + u_N) - x_1, \forall 0 < k < N'; \Lambda_{N'} = (q'N', p'N')) \\ & \geq \mathbf{P}_{\beta,0}(\Lambda_{N'} = (q'N', p'N')) \\ & - \sum_{k=1}^{N'} \mathbf{P}_{\beta,0}(X_k < \tilde{f}(k + u_N) - x_1; \Lambda_{N'} = (q'N', p'N')) \\ & \geq e^{-N'h_{N'}^{q',p'} \cdot (q',p') + \mathcal{L}_{\Lambda_{N'}}(h_{N'}^{q',p'})} \\ & \quad \times \left(\mathbf{P}_{N',h_{N'}^{q',p'}}(\Lambda_{N'} = (q'N', p'N')) - e^{-\lambda\zeta u_N} e^{\lambda(a_1 - K)} \sum_{k=1}^{N'} e^{-C_{20}k} \right), \end{aligned}$$

and $\sum_{k \geq 1} e^{-C_{20}k} < \infty$, which concludes the proof. \square

APPENDIX A: THE WETTING MODEL

In this appendix, we provide well-known results and estimates on the wetting model which are proven in [11]. Recall that we defined \mathbf{P}_β in (2.8), and for all $N \geq 1$, let

$$(A.1) \quad Z_{\text{wet},N}^{\beta,\delta} := \mathbf{E}_{\beta,0}[e^{\delta \sum_{k=1}^N \mathbf{1}_{\{X_k=0\}}} \mathbf{1}_{\{X \in B_N^{0,+}\}}],$$

be the (constrained) partition function of a wetting model associated with a random walk of law \mathbf{P}_β . Let $h_\beta(\delta)$ be its free energy—recall (2.9), and notice that it is well defined because $(Z_{\text{wet},N}^{\beta,\delta})_{N \geq 1}$ is super-multiplicative. Note that $h_\beta(\delta)$ is nondecreasing in δ , that $h_\beta(\delta) \leq \delta$, and recalling (6.1), we have $h_\beta(0) = 0$; hence $h_\beta(\delta) \geq 0$ for all $\delta \geq 0$, and we define $\tilde{\delta}_c(\beta) := \inf\{\delta \geq 0, h_\beta(\delta) > 0\} \in [0, \infty]$.

Let $\tau := \inf\{t \geq 1; X_t \leq 0\}$ and let $K_\beta(t) := \mathbf{P}_{\beta,0}(\tau = t, X_t = 0)$, $t \geq 1$. Notice that conditionally to τ , $-X_\tau$ follows a geometric distribution on \mathbb{N}_0 (in particular τ and X_τ are independent). Therefore, one has $\sum_{t \geq 1} K_\beta(t) = \mathbf{P}_{\beta,0}(X_\tau = 0) = 1 - e^{-\beta/2}$, and $K_\beta(t) = \mathbf{P}_{\beta,0}(\tau = t)(1 - e^{-\beta/2})$. It is a straightforward application of [3], (4.5), that there exists $c > 0$ depending on β only such that

$$(A.2) \quad K_\beta(t) \sim \frac{c}{t^{3/2}} \quad \text{as } t \rightarrow \infty.$$

PROPOSITION A.1 ([11], Theorem 2.2 and equation 2.2).

(i) For $\beta > 0$, the free energy of this wetting model $h_\beta(\delta)$ is the only solution in $\zeta \geq 0$ of the equation

$$(A.3) \quad \sum_{t \geq 1} K_\beta(t) e^{-\zeta t} = e^{-\delta},$$

if it exists (i.e., if $\delta > \tilde{\delta}_c(\beta) = -\log(1 - e^{-\beta/2})$), and 0 otherwise. An explicit expression of $h_\beta(\delta)$ is given by

$$(A.4) \quad h_\beta(\delta) = \log\left(\frac{(e^\delta - 1)(1 - e^{-\beta/2})^2}{1 - e^{-\delta} - e^{-\beta}}\right), \quad \text{for } \delta > -\log(1 - e^{-\beta/2}),$$

and 0 otherwise.

(ii) Recall the definition of c in (A.2). The asymptotics of the partition function are given by:

(a) if $\delta > \tilde{\delta}_c(\beta)$, then as $N \rightarrow \infty$,

$$Z_{\text{wet},N}^{\beta,\delta} \sim C_{\text{wet}} e^{h_\beta(\delta)N} \quad \text{where } C_{\text{wet}} := \left(e^\delta \sum_{t \geq 1} t K_\beta(t) e^{-h_\beta(\delta)t} \right)^{-1},$$

(b) if $\delta < \tilde{\delta}_c(\beta)$, then as $N \rightarrow \infty$,

$$Z_{\text{wet},N}^{\beta,\delta} \sim \frac{c e^{-\delta}}{(e^{-\delta} - 1 + e^{-\beta/2})^2} N^{-3/2},$$

(c) if $\delta = \tilde{\delta}_c(\beta)$, then as $N \rightarrow \infty$,

$$Z_{\text{wet},N}^{\beta,\tilde{\delta}_c(\beta)} \sim \frac{1 - e^{-\beta/2}}{2\pi c} N^{-1/2}.$$

The first claim follows from [11], (2.2), and above observations regarding K_β . The explicit formula for $h_\beta(\delta)$ is obtained by computing

$$\sum_{t \geq 1} K_\beta(t) e^{-\zeta t} = (1 - e^{-\beta/2}) \mathbf{E}_{\beta,0}[e^{-\zeta \tau}] = 1 - e^a e^{-\beta/2},$$

for some $a > 0$, thanks to a stopping time argument and with the observation that $(e^{-aX_n - \zeta n})_{n \geq 0}$ is a martingale when $a > 0$ is the unique solution of $\mathbf{E}_{\beta,0}[e^{-aX_1 - \zeta}] = 1$; details are left to the reader. The asymptotics of $Z_{\text{wet},N}^{\beta,\delta}$ as $N \rightarrow \infty$ are given in [11], Theorem 2.2, where we recall (A.2).

APPENDIX B: AN FKG INEQUALITY

We provide here a (conditional) FKG inequality for random walks of length $N \in \mathbb{N}$ with increment distributed as \mathbf{P}_β , similarly to more general, already well-known FKG inequalities as developed in [25]. Define a partial order on \mathbb{Z}^N : for all $u, v \in \mathbb{Z}^N$, $u \preceq v$ if and only if $u_k \leq v_k$ for all $1 \leq k \leq N$. A function $f : \mathbb{Z}^N \rightarrow \mathbb{R}$ is nondecreasing (resp. nonincreasing) if for all $u, v \in \mathbb{Z}^N$, $u \preceq v$ implies $f(u) \leq f(v)$ (resp. $f(u) \geq f(v)$). Define also for every $u, v \in \mathbb{Z}^N$,

$$(B.1) \quad \begin{aligned} u \wedge v &:= (\min(u_1, v_1), \dots, \min(u_N, v_N)), \\ u \vee v &:= (\max(u_1, v_1), \dots, \max(u_N, v_N)). \end{aligned}$$

PROPOSITION B.1. Let $A \subseteq \mathbb{Z}^N$ be such that:

- $\mathbf{P}_{\beta,0}(A) > 0$,
- for any $u, v \in A$, one has $u \wedge v \in A$ and $u \vee v \in A$.

Let $f, g : A \rightarrow \mathbb{R}$ be two nonincreasing (or two nondecreasing), nonnegative and bounded functions. Then,

$$(B.2) \quad \mathbf{E}_{\beta,0}[f(X)g(X)|X \in A] \geq \mathbf{E}_{\beta,0}[f(X)|X \in A] \mathbf{E}_{\beta,0}[g(X)|X \in A].$$

Note that this claim holds in particular for $A = B_N^{0,+}$ or $A = \mathbb{Z}^N$.

PROOF. We prove the proposition for f, g both nonincreasing. Let $u, v \in A$. For every subset $B \subseteq A$, let us define $\mu_1(B) := \mathbf{P}_{\beta,0}(B|A)$, and

$$\mu_2(B) := \frac{\mathbf{E}_{\beta,0}[g(X)1_B(X)|X \in A]}{\mathbf{E}_{\beta,0}[g(X)|X \in A]},$$

where we assume without loss of generality that $\mathbf{E}_{\beta,0}[g(X)|X \in A] > 0$ —otherwise $g \equiv 0$ on A and the proposition is trivial. Notice that μ_1 and μ_2 are probability measures on A . To apply an FKG inequality from [25], we must prove that for any $u, v \in A$,

$$(B.3) \quad \mu_1(u \vee v)\mu_2(u \wedge v) \geq \mu_1(u)\mu_2(v).$$

To that end we multiply the left-hand side in (B.3) by $\mathbf{E}_{\beta,0}[g(X)|A] \times \mathbf{P}_{\beta,0}(A)^2$, to obtain

$$(B.4) \quad \begin{aligned} & \mathbf{E}_{\beta,0}[g(X)|A]\mathbf{P}_{\beta,0}(A)^2\mu_1(u \vee v)\mu_2(u \wedge v) \\ &= g(u \wedge v)\mathbf{P}_{\beta,0}(X = u \vee v)\mathbf{P}_{\beta,0}(X = u \wedge v) \\ &= g(u \wedge v)c_\beta^{-2N} \prod_{k=1}^N e^{-\frac{\beta}{2}|(u_k \vee v_k) - (u_{k-1} \vee v_{k-1})|} \prod_{k=1}^N e^{-\frac{\beta}{2}|(u_k \wedge v_k) - (u_{k-1} \wedge v_{k-1})|}, \end{aligned}$$

where $u_0 = v_0 := 0$. Moreover, we claim that for any $a, b, c, d \in \mathbb{R}$,

$$(B.5) \quad |a \vee b - c \vee d| + |a \wedge b - c \wedge d| \leq |a - c| + |b - d|.$$

To prove this, observe that $|a - c| = a + c - 2(a \wedge c)$ for any $a, c \in \mathbb{R}$, hence (B.5) is equivalent to

$$a \wedge c + b \wedge d \leq (a \vee b) \wedge (c \vee d) + (a \wedge b) \wedge (c \wedge d).$$

Notice that $(a \wedge b) \wedge (c \wedge d) = \min\{a, b, c, d\}$ is either equal to $a \wedge c$ or $b \wedge d$. Assume $a \wedge b \wedge c \wedge d = a \wedge c$ without loss of generality. Then one obviously has $b \wedge d \leq (a \vee b) \wedge (c \vee d)$, which concludes the proof of (B.5).

Using (B.5) in (B.4) and recalling that g is nonincreasing, we conclude

$$\begin{aligned} & \mathbf{E}_{\beta,0}[g(X)|A]\mathbf{P}_{\beta,0}(A)^2\mu_1(u \vee v)\mu_2(u \wedge v) \\ & \geq g(v)c_\beta^{-2N} \prod_{k=1}^N e^{-\frac{\beta}{2}|u_k - u_{k-1}|} \prod_{k=1}^N e^{-\frac{\beta}{2}|v_k - v_{k-1}|} \\ & \geq \mathbf{E}_{\beta,0}[g(X)|A]\mathbf{P}_{\beta,0}(A)^2\mu_1(u)\mu_2(v), \end{aligned}$$

which eventually proves (B.3).

Since f is also a nonincreasing function on A , we can use a generalized FKG inequality claimed in [25], Theorem 3, to obtain

$$(B.6) \quad \int_A f d\mu_1 \leq \int_A f d\mu_2,$$

which concludes the proof. \square

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